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MODESTUM

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A cognitive approach to concept of visualization in problem-solving

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ABSTRACT

Any ball or small stone that is thrown follows a certain path. Many of us have experienced throwing objects towards a specific goal many times. If we throw a ball to someone or against something, the ball will follow a certain path, a visualized motion. Hopefully you can imagine a motion path if you are asked to throw something towards a target some 10 meters away. Somehow we can imagine the height and path in order to get the ball (or small stone) in the basket or what we are aiming at. If we want to shoot a basketball to go into the basket, we consider elevation angle and initial velocity also. Most of us humans do that without thinking about it.

Keywords: visualization, substritution, problem-solving

INTRODUCTION

To be able to visualize mathematical problems is an important ability. How visualization help us learning mathematics is still unclear although many research studies. Many students meet visualization techniques already when meeting arithmetic word problems in elementary school.

Problem 1

Jennifer has four apples and Helena give Jennifer two more apples. How many apples has Jennifer then?

This situation is possible to visualize in different ways and I have chosen the method in **Figure 1** with GeoGebra as the technology.

But there are mathematical problems, where the visualization is difficult to achieve. I have always noted what I am supposed to do with an example of a mathematical problem.

Problem 2

Among any sequence of 12 consecutive and positive numbers there is one of these numbers that is smaller than the sum of its proper divisors.

We can probably see the 12 consecutive and positive numbers, but the divisors are harder to see.

Solution

At least one of these 12 consecutive and positive numbers is a multiple of 12 and we call it $12n$.

The proper divisors to $12n$ are $2n$, $3n$, $4n$, and $6n$. Compare it to the divisors for 12.

The sum of the divisors is $2n+3n+4n+6n=15n$, which is larger than $12n$.

Conclusion

This problem and other problems about the same are perhaps impossible to visualize but I have enjoyed writing what I am supposedly to do with the problem, and it seems to me that I understands more when I do my writing.

Existence

In mathematics, an existence theorem is a theorem that claim the existence of a certain object. It can be, and often is, a statement beginning with the statement “there exist” (**Figure 2**).

We see that there are six equal triangles inside the large triangle so there are altogether seven triangles in **Figure 2**. What will happen if we take away triangle 1 and triangle 2? Will triangle 4 just disappear and stop to exist?

Figure 1. GeoGebra helps us organize apples (Source: Author's own elaboration)

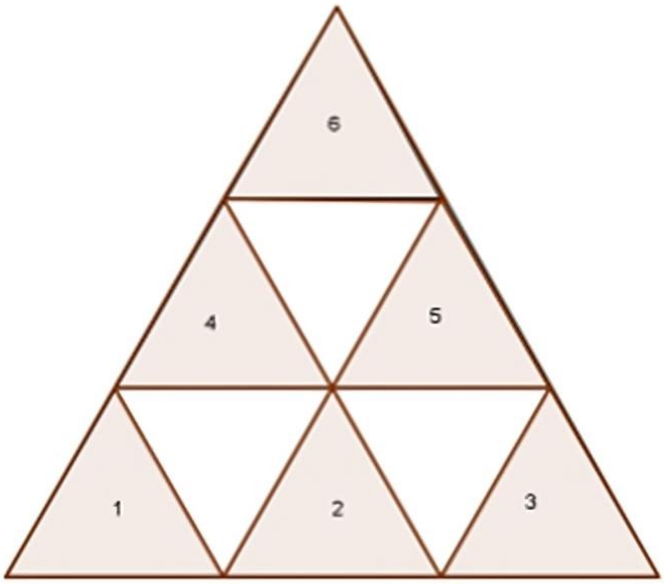

Figure 2. Triangles (Source: Author's own elaboration)

THEORETICAL FRAMEWORK

A limit value or a border value can exist outside mathematics as well. Even young students know something about borders between countries or perhaps between houses or city districts. In an environmental discussion may the concept of border values be something to discuss in relation to both humans and environmental sustainability.

Students in elementary school learn that the sequence of natural numbers 1, 2, 3, 4, 5, 6, … has no upper limit. Thus, students have some understanding of the concept of infinity. Mathematical defined limit values are often defined in relation to a number or a sequence that approaches infinity. Just as many other mathematical concepts, infinity may be seen as both a process, as when we are approaching infinity in the principle of induction, or as an object, as a very large number or as the cardinality of a set. Process-object duality in mathematical thinking is a well-established field of research. Classical papers are by Gray and Tall (1994), Breidenbach (1991), and Sfard (1992). Infinity is an interesting aspect of mathematics to be discussed and analyzed.

One example is Euclid's proof of the existence of an infinity of prime numbers, a good example of the very fundament of natural numbers from the book *A mathematician's apology* (Hardy, 2004 [1940]).

> The prime numbers or primes are the numbers (A) 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, …, which cannot be resolved into smaller factors. Therefore also 37 and 317 are prime. The primes are the material out of which all numbers are built up by multiplication: thus 666=2×3×3×37. Every number, which is not prime itself is divisible by at least one prime (usually, of course, by several). We have to prove that there are infinitely many primes, i.e., that the series (A) never comes to an end.
>
> Let us suppose that it does, and that 2, 3, 5, … , P is the complete series (so that P is the largest prime); and let us, on this hypothesis, consider the number Q defined by the formula Q=(2×3×5× …×P)+1.
>
> It is plain that Q is not divisible by and of 2, 3, 5, …, P; for it leaves the remainder one when divided by any one of these numbers. But, if not itself prime, it is divisible by some prime, and therefore there is a prime (which may be Q itself) greater than any of them. This contradicts our hypothesis, that there is no prime greater than P; and therefore, this hypothesis is false.
>
> The proof is by *reductio ad absurdum*, and *reductio ad absurdum*, is one of a mathematician's finest weapons. It is a far finer gambit than any chess gambit: A chess player may offer the sacrifice of a pawn or even a piece, but a mathematician offers *the game* (Hardy, 2004 [1940], p. 17-18).

Young students also early learn about how universe is being limitless large and atomic structures being limitless small. This is of course not easy to understand. Is seems as if most students have extreme difficulties to obtain a reliable and robust image of the limit concept, but it is hard to find solutions to this situation (Mamona-Downs, 2001).

> The problem lies largely in the surprisingly rich intuitive base that many students seem naturally to be endowed with in the theme. We say surprisingly because infinity is something never directly experienced by the senses in the physical world. However, there appears to be a common part of the human psyche that directs the human brain to contemplate infinity, in particular the unbounded universe and the infinitely small. This carries through to the notion of limit (p. 259).

It seems convincing that we humans construct our understanding of infinity and limit in different ways. Tall (2004) suggests a possible categorization of cognitive growth into three worlds of mathematics or three distinct but interacting developments. Three worlds of mathematics are founded on the assumption that the learning of mathematical concepts is individual and develops at different stages: through perception, through symbols or axioms.

The first world is the conceptual-embodied world, the world we meet through perception, the visual and spatial mathematical world. Most of us have a concept image of a circle. A circle is round, it may be large or small, and it may be red or blue. We have not learned this through educational efforts; instead, we have learned this through the physical world and through observations. The first mathematical world consists of objects we have discovered and observed in the real world, knowledge we have gained through our senses. It also contains mental conceptions of non-existing objects such as a point with no size and lines with no thickness. This is in coherent with Lakoff and Nunez's (2000) claims that mathematics draws on and is founded in bodily experience.

The second world is the perceptual–symbolic world. In this world, we find symbols and actions that we have to perform when we, for example, are dealing with manipulations in algebra. In this mathematical world is the concept of procept central, which consists of the first part of process and the end of the word concept. Gray and Tall (1994) introduced the concept procept to describe a central part of the learning of mathematical concepts. Gray and Tall (1994) underlined that it is important to learn how to apprehend mathematical symbols both as concepts and as parts of a process at the same time.

> An elementary procept is the amalgam of three components: A process, which produces a mathematical object, and a symbol, which is used to represent either process or object (Gray & Tall, 1994, p. 12).

According to Gray and Tall (1994), the symbolic structure 2×3 may be perceived as a process (multiplication) or as a concept (product).When the individual is in this symbolic world, he/she may use and reflect over the mathematical symbolic language and its function, meaning and application.

The third mathematical world is the formal mathematical world. Is this world we find axioms, theorems and proofs in focus. Based on given assumptions regarding the proportion and relation between mathematical objects are axiom-based structures built and used as foundations for mathematical theorems. Mathematical thinking is thereby based on perception developing subtly in sophistication through the mental world of conceptual embodiment, operations developing through actions that become mathematical operations in a world of operational symbolism and increasingly subtle use of verbal reasoning that leads to formal aspects of embodiment and symbolism and, eventually, to a world of axiomatic formalism. The development takes account of the individual's previous experience, which may operate successfully in one context yet remain supportive or become problematic in another, giving rise to emotional reactions to mathematics, leading to a spectrum of success and failure over the longer term (Tall, 2004).

The theories about cognitive development of mathematical knowledge articulated in Tall and Vinner (1981) and Gray and Tall (1994) are in many ways quite comparable with the historical development of mathematics as an axiomatic science.

Tall (2000) furthermore discuss the concept of procept in relation to limits, as follows:

> Different kinds of procept bring different cognitive challenges. Experiences with arithmetic symbols, with built-in processes of computation, can lead students to find algebraic expressions difficult to contemplate; they find themselves being asked to manipulate symbols that represent processes, which cannot themselves be evaluated (unless the numerical values of the variables are known). Limit procepts have a potentially infinite process of evaluation and carry a sense of "getting close", "getting large", or "getting small"; this can lead either to a sense of a limit never being reached or to the conception of a number line with infinitesimal and infinite quantities (p. 216-217).

If we study a typical limit value question in the middle of secondary school, we will see that the learning and conceptualizing that takes place is perhaps even more complicated than that.

Let us have a look at the limit of $\lim_{x\to\infty}\sqrt{\frac{16x}{4x+9}}$ $\lim_{x\to\infty}\sqrt{\frac{16x}{4x+9}}$. For students with a good understanding of algebra and arithmetic, it is possible to divide all terms with x and to draw the conclusion that $9/x$ will go to zero when x goes to infinity and therefore $\lim_{x\to\infty}\sqrt{\frac{16x}{4x+9}} = \pm 2$. $\text{limes}_{x\to\infty}\sqrt{\frac{16x}{4x+9}}$ That procedure has little to do with the concepts of infinity apart of knowing something about dividing with large numbers and also believing that $9/x$ will go to zero. If we on the other hand allow a technological tool to sketch the curve, we could arrive to something like this in **Figure 3**.

We could probably convince someone that this curve would never go over the two value, but we cannot see it. No matter of how much we try, we cannot get the tool to sketch $f(x)$ when x approach infinity. In any of these two methods, we do not see what

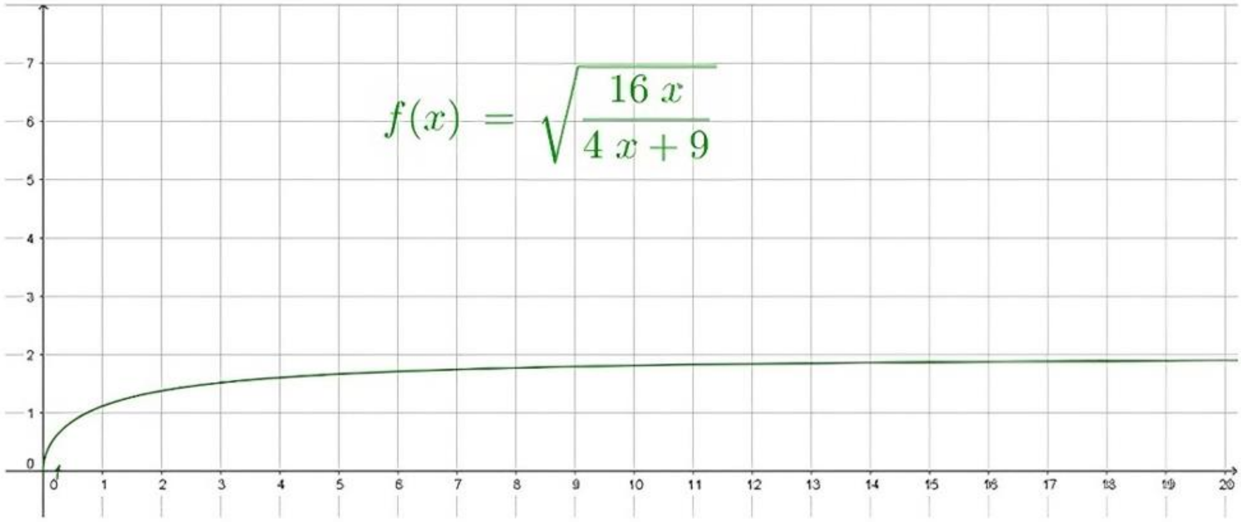

Figure 3. GeoGebra helps us see a sketch of a limit (Source: Author's own elaboration)

happens when *x* approach infinity. We just believe that something is happening. We translate and use what we have nearby, and we stretch that out to infinity wherever that is. It seems as if the visualization is both beneficial and dangerous for our mind.

There are also mathematical problems that want us to compare two or more values.

Which is the largest number of two or $(1.001)^{1,000}$?

We see that $(1.001)^{1,000} = (1 + \frac{1}{1,000})^{1,000}$ and we know that $(1 + \frac{1}{n})^n = e$.

Solution strategy: We see that $(1.001)^{1,000} = (1 + \frac{1}{1,000})^{1,000}$ and we recognize $(1 + \frac{1}{n})^n$. Let us use the binomial theorem:

$(a + b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b^1 + \cdots + \binom{n}{n}b^n$

$$(1 + \frac{1}{1,000})^{1,000} = \binom{1,000}{0} + \binom{1,000}{1}(\frac{1}{1,000})^1 + \binom{1,000}{2}(\frac{1}{1,000})^2 + \cdots + \binom{1,000}{1,000}(\frac{1}{1,000})^{1,000}$$

$$(1 + \frac{1}{1,000})^{1,000} = 1 + 1 + \binom{1,000}{2}\frac{999}{1,000^2} + \cdots + 1 \times \frac{1}{1,000^{1,000}}$$

$$(1 + \frac{1}{1,000})^{1,000} = 2 + \frac{999}{2,000} + \cdots + 1 \times \frac{1}{1,000^{1,000}}$$

All numbers are >0.

So, $(1 + \frac{1}{1,000})^{1,000} > 2$.

In fact, $\lim_{x \to \infty}(1 + \frac{1}{n})^n = e$.

Geometrical problems are naturally to visualize. They can still be complicated, though.

A rectangle is inscribed in the ellipse $\frac{x^2}{4} + y^2 = 1$.

What are the dimensions of the rectangle? Which is the maximal area?

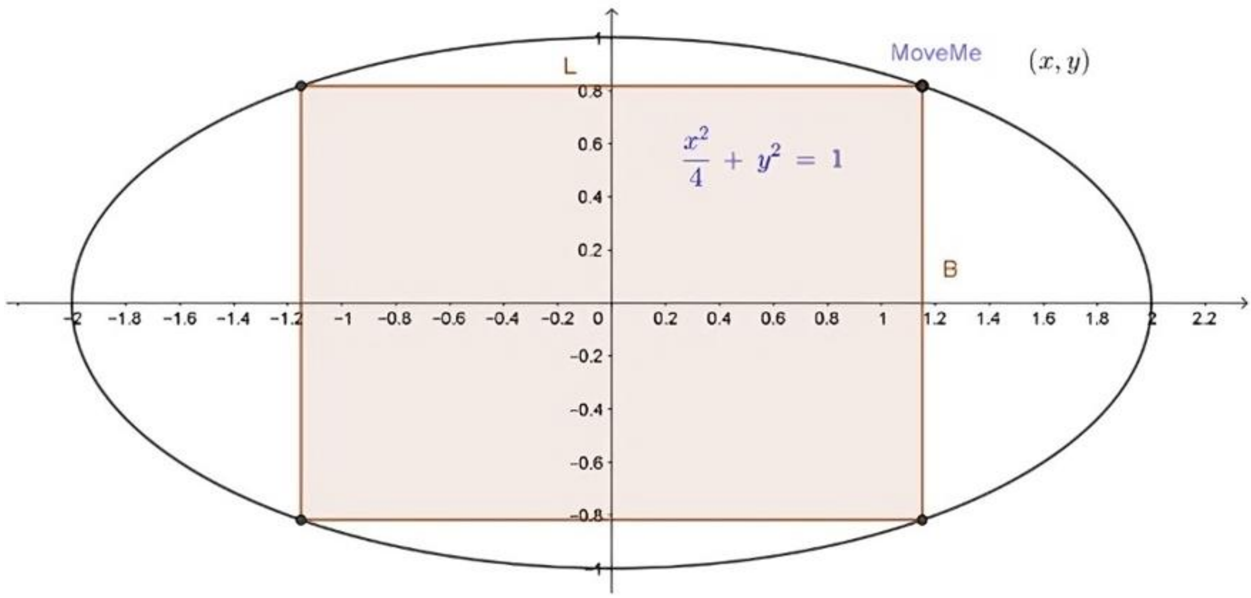

Figure 4. GeoGebra shows us a rectangle inscribed in an ellipse (Source: Author's own elaboration)

Solution strategy: We use GeoGebra to get at graph of the situation (**Figure 4**).

Solution: The area of the rectangle is A=*L*×*W*. We label this area with *x* and *y*.

We have that (*x*, *y*) is the corner in the first quadrant at the inscribed rectangle. It gives that the length in the first and second quadrant can be written as *L*=2*x* and the height can be written as *W*=2*y*. We get that the area=2*x*·2y=4xy. Since the ellipse is $x^2/4+y^2=1$ $\mathbf{\frac{x^2}{4} + y^2 = 1}$ and y>0 we can write $y = \sqrt{1-\frac{x^2}{4}}$.

$Area(x) = 4x\sqrt{1-\frac{x^2}{4}} = 2x\sqrt{4-x^2}$.

$Area'(x) = -\frac{4(x^2-2)}{\sqrt{4-x^2}}$. The derivative is obviously zero when $x = \sqrt{2}$.

The maximum area is thus $Area(\sqrt{2}) = 2\sqrt{2}\sqrt{4-2} = 2\sqrt{2}\sqrt{2} = 2 \times 2 = 4$.

See Lingefjärd (2022) for a discussion of this and some similar problems.

A special type of problems is when we are asked to simplify an expression. We often do that by using algebra. A simplification can also be to calculate something.

Simplify the following expression: $\sqrt{1+98\times 99\times 100\times 101}$

The solution strategy is to first use substitution. See Lingefjärd (2023).

Substitute $x = 100$.

$\sqrt{1+(x-2)\cdot(x-1)\cdot x\cdot(x+1)}$

Simplify and rewrite, as follows:

$\sqrt{1+x\cdot(x-1)\cdot(x-2)\cdot(x+1)}$

$\sqrt{1+(x^2-x)\cdot(x^2+x-2x-2)}$

$\sqrt{1+(x^2-x)\cdot(x^2-x-2)}$

Substitute $y = x^2 - x$

$\sqrt{1+y\cdot(y-2)}$

Simplify and rewrite, as follows:

$\sqrt{y^2-2y+1}$

$\sqrt{(y-1)^2} = y-1 => x^2-x-1$

$x = 100$

$x^2 - x - 1$

$100^2 - 100 - 1$

$10{,}000 - 100 - 1$

$9{,}900 - 1 = 9{,}899$

Thus, $\sqrt{1+98\times 99\times 100\times 101} = 9{,}899$

A type of problems that combine algebra, geometry and solution strategies is integral problems.

Let us have a look at an integral problem from Lingefjärd (2022).

Solve the following integral: $\int_0^1 \frac{-x^2+1}{x+\sqrt{-x^2+1}}dx$

Our solution strategy: Substitute with trigonometric identities!

$x = \sin\theta\,, dx = \cos\theta d\theta$

$x = 0 \rightarrow \sin\theta = 0$

$x = 1 \rightarrow \sin\theta = \frac{\pi}{2}$

$$\int_0^{\frac{\pi}{2}} \frac{(1-\sin^2\theta)(\cos\theta)d\theta}{\sin\theta + \sqrt{1-\sin^2\theta}} = \int_0^{\frac{\pi}{2}} \frac{(\cos^2\theta)(\cos\theta)d\theta}{\sin\theta + \sqrt{\cos^2\theta}} = \int_0^{\frac{\pi}{2}} \frac{(\cos^3\theta)d\theta}{\sin\theta + \cos\theta}$$

Substitute $\theta = \frac{\pi}{2} - u$ and beware of the limit values.

$$\int_{\frac{\pi}{2}}^0 \frac{(\sin^3 u)(-du)}{\cos u + \sin u} = \int_0^{\frac{\pi}{2}} \frac{(\sin^3 u)(du)}{\sin u + \cos u}$$

We have that.

$$\int_0^{\frac{\pi}{2}} \frac{(\sin^3 u)(du)}{\sin u + \cos u} = \int_0^{\frac{\pi}{2}} \frac{(\sin^3\theta)d\theta}{\sin\theta + \cos\theta}$$

We know that $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

The integral $I = \frac{1}{2}(I + I)$

We get $\frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{(\sin^3\theta)+(\cos^3\theta)(d\theta)}{\sin\theta+\cos\theta} = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{((\sin\theta)+(\cos\theta))(((\sin^2\theta)-(\sin\theta\times\cos\theta))+\cos^2\theta)(d\theta)}{\sin\theta+\cos\theta}$

$$\frac{1}{2}\int_0^{\frac{\pi}{2}} (((\sin^2\theta) - (\sin\theta \times \cos\theta)) + \cos^2\theta)d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}}\left[1 - \frac{1}{2}\sin 2\theta\right] = \frac{1}{2}\left[0 + \frac{\cos 2\theta}{4}\right]_0^{\frac{\pi}{2}} = \frac{1}{2}\left[\left(\frac{\pi}{2} + \frac{-1}{4}\right) - \left(0 + \frac{1}{4}\right)\right]_0^{\frac{\pi}{2}} = \frac{1}{2}\left[\frac{\pi}{2} - \frac{1}{2}\right] = \frac{\pi-1}{4}$$

$$\frac{1}{2}\cdot\int_0^{\frac{\pi}{2}} \frac{(\sin^3(\theta)+\cos^3(\theta))(d\theta)}{\sin(\theta)+(\cos(\theta))} = \frac{1}{2}\cdot\int_0^{\frac{\pi}{2}} \frac{(\sin(\theta)+(\cos(\theta))(\sin^2(\theta)-(\sin(\theta)\cdot(\cos(\theta))+\cos^2(\theta))(d\theta)}{\sin(\theta)+(\cos(\theta))}$$

Approximative value of $\frac{\pi-1}{4} = 0.53539816339744830$.

Let me present you for one last mathematical problem to deal with.

Solve $7^{(4x-5)} = 3^{(2x+1)}$.

Solution strategy: Logarithms since we have exponents.

$\log 7^{(4x-5)} = \log 3^{(2x+1)}$

$(4x - 5)\log 7 = (2x + 1)\log 3$

$4x \cdot \log 7 - 5\log 7 = 2x \cdot \log 3 + \log 3$

I will now use the method of gathering together the terms with multiplication of *x* to the left side and the terms without *x* to the right side. We do that because we like to have *x* alone to the left side.

$4x \cdot \log 7 - 2x \cdot \log 3 = 5 \cdot \log 7 + \log 3$

$x \cdot (4 \cdot \log 7 - 2\log 3 = 5 \cdot \log 7 + \log 3)$

$$x = \frac{(5\cdot\log 7 + \log 3)}{(4\cdot\log 7 - 2\cdot\log 3)} = 1.938302302825\cdots$$

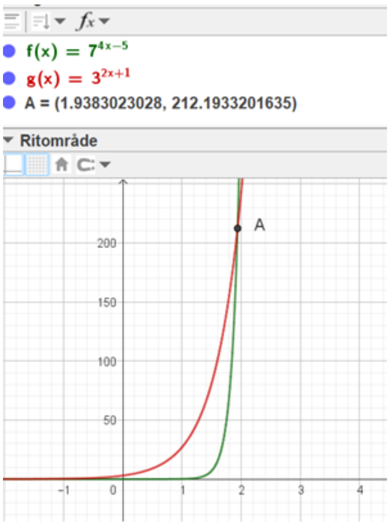

Figure 5. GeoGebra can show us curves of complicated algebraic symbolism (Source: Author's own elaboration)

See how GeoGebra can "solve" this problem by finding the interception point by these two curves (**Figure 5**).

CONCLUSIONS

Please notify that I have only showed you some problems of all mathematical problems that exist. All these problems are difficult the first time you see them and slowly they become as old acquaintances. Nevertheless, there is only one way to become a good problem solver and that is to solve problems and enjoy it at the same time. During my long experience of solving mathematical and physical problems I have benefitted from a graphical view of some of the problem. When I look back at solutions I did when I was young I note that I actually wrote "let us substitute" in the margin in some of the problems.

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