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MODESTUM

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A cognitive aspect on substitution in problem solving in mathematics

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ABSTRACT

This is an article about the methodology of substitution in problem solving of mathematics. The operative use of substitution depends on two things: in an equation or situation in which variables appear, a substitution is a change of variable, and second, the change of variable is effective only if the change of variable makes the expression easier to understand and makes it possible to solve the problem. Substitution will be successful if the original equation has a symmetry or especial property that is possible to explore, and the proficiency in using the method of substitution depends on this.

Keywords: algebra, problem solving, substitution

INTRODUCTION

Problem solving exist both as a central concept and as a domain in curriculum documents around the world. I have been thinking about and teaching problem solving for about 40 years.

A prototype for problem solving was Pólya’s (1945) classical schema with four steps for problem solving:

(1) To understand the problem

(2) To make a plan (or find a mathod)

(3) To use the plan

(4) Looking back (evaluation)

Pólya’s (1945) heuristic method is not sprung from a theory of learning and therefore not a schema for teaching problem solving.

Problem 1

A problem from my early years in school.

You are about to plant a garden with five rows of trees with four trees in every row. Due to poor economy, there are only 10 trees available.

Solution

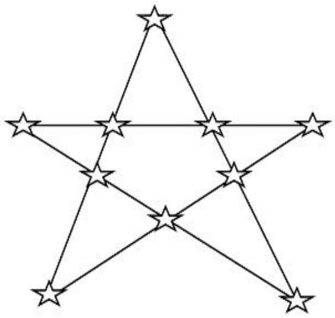

Figure 1. Ten trees (Source: Created by the author in GeoGebra)

Problem 2

Another problem from compulsory school.

A car is driven with a velocity of 60 km/h from A to B and with the velocity of 40 km/h back from B to A. What is the average velocity from the whole travel back and forth?

Harmonical mean gives the following: $\frac{2a_{1a_2}}{a_1+a_2} = \frac{2(60\cdot 40)}{(60+40)} = 48\ km/h.$

PROBLEM SOLVING AND SUBSTITUTION

This is not an article about how students behave when they solve problems. There is an almost infinity number of articles about that. This is an article about the problems we solve with substitution. An important method in problem solving is substitution, as a way to reveal the cognitive effort and internal stress caused by mathematical problem solving. In substitution we use symbols more well-known to our mind so that they can reveal our brain and perhaps open up more familiar methods for us to use. Methods as addition, subtraction, or perhaps will we minimize the problem solving process to solve quadratic second degree equations.

"How should I be able to see that?" This is a natural response when viewing a substitution in mathematics, and this article attempts to answer this question. Unfortunately, the method or technique of substitution is often presented without mentioning the idea behind substitutions. The operative use of substitution depends on two things: in an equation or situation in which variables appear, a substitution is a change of variable, and second, the change of variable is effective only if the change of variable makes the expression easier to understand and makes it possible to solve the problem.

Substitution will be successful if the original equation has a symmetry or especial property that is possible to explore, and the proficiency in using the method of substitution depends on this. Therefore, we should search for special features of the problem and then be ready to substitute and explore the changes the substitution gives. If we are able to solve the problem in the new variables the substitution gives, we should rewrite the solution in the original variables.

In problem-solving by substitution, mathematical operators are sometimes substituted with symbols. We are given some expression, or equation or graph involving the variable *x*. We make the substitution $x=f(t)$, and so we have a new expression, equation or graph involving the given terms, the function *f* and the variable *t*. Since we are free to choose *f* to be any function, hopefully the choice of *f* in the new expression in *t* is easier than the original expression in *x*. The proficiency is in the selection of *f* since the rest is just the algebraic manipulation of the variables. In general, you should feel that you make the problem easier to solve with the substitution you provide.

Problem 3

How should we solve the following problem?

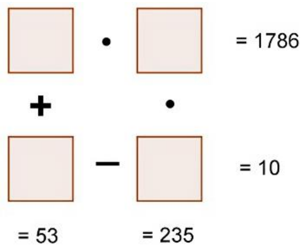

Figure 2. A problem in boxes (Source: The author)

Solution strategy: Substitution

Let us call the boxes *a*, *b*, *c*, and *d*.

- $a \cdot b = 1{,}786$ (1)
- $b \cdot d = 10$ (2)
- $a + c = 53$ (3)
- $c - d = 10$ (4)

Notify the difference between this notation and the boxes in **Figure 2**. The algebraic notation informs us that there are four unknown variables but also four equations. The problem is thus solvable, and we can identify which equations we would like to start with, is a result from the substitution.

We subtract (3) – (4) and we get,

$a + d = 43$ (5)

Equation 1 and equation 2 gives that $b = \frac{1{,}786}{a}$ and $b = \frac{235}{d}$. It gives that $b = \frac{1{,}786}{a} = \frac{235}{d}$ and we get (divide with 47) that $38d=5a$. Let us use that $38d = 5a$ in equation (5).

$a + d = 43$

$38d = 5a$

We multiply equation (5) with 5.

$5a + 5d = 215$

$38d + 5d = 215$

$d = \frac{215}{43} = 5.$

Thus, $b = 235/5 = 47$ and $a = 1{,}786/47 = 38$. Since $c - 5 = 10$, $c = 15$.

Solution

$a = 38$

$b = 47$

$c = 15$

$d = 5$

In my view, this problem became easier to solve with the substitution to algebraic reasoning.

Substitution is however more useful and important for problems at the university level although many of these problems also can be presented at the upper secondary school level.

Problem 4

Solve the equation $3 \cdot \sqrt{\log(x)} + 2 \cdot \log \sqrt{x^{-1}} = 2$.

A solution method is to use the logarithm laws, rewrite, and use substitution. We will once again arrive to a second-degree equation.

The first logarithm law: $\log\ (x \cdot y) = \log x + \log y$

The second logarithm: $\log\left(\frac{x}{y}\right) = \log x - \log y$

The third logarithm law: $\log x^a = a \cdot \log x$

$\log x^a = a \cdot \log x$

We use the second and third logarithm law.

$3 \cdot \sqrt{\log(x)} + 2 \cdot \log \sqrt{x^{-1}} = 2$

$3 \cdot \sqrt{\log(x)} + \log\left(\sqrt{x^{-1}}\right)^2 = 2$

$3 \cdot \sqrt{\log(x)} + \log\frac{1}{x} = 2$

$3 \cdot \log(x)^{1/2} - \log(x) - 2 = 0$

Solution strategy: substitution

$y = \log(x)^{1/2}$

$y^2 = \log(x)$

$3 \cdot y - y^2 - 2 = 0$

$y^2 - 3y + 2 = 0$

$(y-1)(y-2) = 0$

We see that $y = 1$ and that $y = 2$ are solutions.

$\log(x)^{\frac{1}{2}} = 1$ gives that $x = 10$

$\log(x)^{\frac{1}{2}} = 2$ gives that $x = 10^4$

Problem 5

Calculate $\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6\ldots}}}}$.

Solution strategy is substitution.

$t = \sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6\ldots}}}}$

We get that $t = \sqrt{6+t}$

$t^2 = 6 + t$

$t^2 - t - 6 = 0$

Once again, the substitution gives us a second-degree equation to solve. Observe that it makes the problem as such easier. Since 6 = 2 · 3 we can rewrite,

$t^2 - 3t + 2t - 6 = 0$

$(t-3)\cdot(t+2) = 0$

Two solutions but only $t = 3$ is correct.

$\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6\ldots}}}} = 3.$

Problem 6

Solve the following equation system:

$a\sqrt{a} + b\sqrt{b} = 189$

$a\sqrt{b} + b\sqrt{a} = 180$

Solution strategy: Substitution twice.

$\sqrt{a} = x => a = x^2$

$\sqrt{b} = y => b = y^2$

$x^3 + y^3 = 189 => (x+y)(x^2 - xy + y^2) = 189$

$x^2y + y^2x = 180 => (xy)(x+y) = 180$

We divide and we get,

$\frac{(x+y)(x^2-xy+y^2)}{(xy)(x+y)} = \frac{189}{180}.$

$\frac{(x^2-xy+y^2)}{(xy)} = \frac{21}{20}.$

Cross wise multiplication gives

$20x^2 - 41xy + 20y^2 = 0$

Divide every term with y^2, and get

$20\frac{x^2}{y^2} - 41\frac{x}{y} + 20 = 0$

Substitute once again with $t = \frac{x}{y}$ and we get,

$20t^2 - 41t + 20 = 0$

Factorise and,

$(4t-5)\cdot(5t-4) = 0.$

$t = \frac{5}{4}$ or $t = \frac{4}{5}$ so $x = 5$ and $y = 4$ or $x = 4$ and $y = 5$.

Since $a = x^2$ we get that $a = 25$ or $a = 16$.

Since $b = y^2$ we get that $b = 16$ or $b = 25$.

The solution set is $\{a, b\} = \{25, 16\}$.

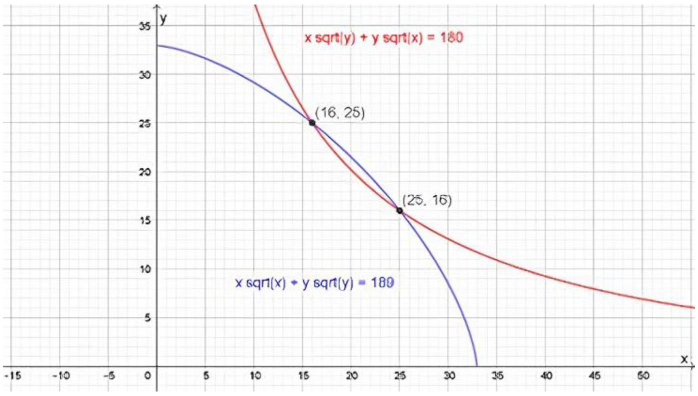

Figure 3. Solution set for the problem 6 (Source: The author)

Problem 7

What number is the largest of 99^{100} or 100^{99}?

Solution strategy: Substitution.

$a = 99^{100}$, $b = 100^{99}$

$\ln(x) = log_e(x)$

We get $\ln(a) = \ln\,(99^{100})$ and $\ln(b) = \ln\,(100^{99})$.

$\ln(a) = 100 \cdot \ln\,(99)$ and $\ln(b) = 99 \cdot \ln\,(100)$.

Let us look at $y\ = \frac{\ln\,(x)}{x}$ and its derivative.

$$y' = \frac{\frac{1}{x} \cdot \mathrm{x} - 1 \cdot \ln(x)}{x^2} = \frac{1 - \ln(x)}{x^2} = 0$$

We get $1 - \ln(x) = 0 \Rightarrow \ln(x) = 1$ and that $\boldsymbol{x = e}$.

x		e	
y'	+	0	
y	↗		↘

There is a maximum for $x = e \Rightarrow y = \frac{\ln x}{x} = \frac{\ln e}{e} = \frac{1}{e}$.

We see that for $x > e$ is the function $y = \frac{\ln\,(x)}{x}$ decreasing.

Obviously, $100 > 99 > e$.

Therefore, $\frac{\ln\ 100}{100} < \frac{\ln\ 99}{99} \Leftrightarrow 99 \cdot \ln\,(100) < 100 \cdot \ln\,(99)$.

Thus, $99^{100} > 100^{99}$.

A general result is that if $x \geq e$, then we have that $x^{x+1} > (x+1)^x$.

And $\frac{99^{100}}{100^{99}} = 36.60324\ldots$

Problem 8

Solve $\int x\sqrt{x^2+1}\,dx$.

Solution strategy: Substitution. Quite many integrals are impossible to solve with paper and pencil without substitution.

$t = x^2 + 1$

$dt\ = 2xdx$

$$dx = \frac{dt}{2x}$$

$$\int x\sqrt{x^2+1}\,dx = \int x \cdot t^{1/2}\frac{dt}{2x} = \frac{1}{2}\int t^{1/2}\,dt$$

$$= \frac{1}{2}\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) = \frac{1}{3}(x^2+1)^{3/2} + C$$

Problem 9

Simplify $\sqrt{\frac{11^4+100^4+111^4}{2}}$.

Solution strategy: Substitution and simplification.

Define $a = 11$ and $b = 100$.

$$\sqrt{\frac{11^4 + 100^4 + 111^4}{2}} = \sqrt{\frac{a^4 + b^4 + (a+b)^4}{2}} = \sqrt{\frac{a^4 + b^4 + a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4}{2}}$$
$$= \sqrt{\frac{2a^4 + 2b^4 + 4a^3b + 6a^2b^2 + 4ab^3}{2}} = \sqrt{a^4 + b^4 + 2a^3b + 3a^2b^2 + 2ab^3}$$

We have that

$(a^2 + b^2 + ab)^2 = a^4 + b^4 + 2a^3b + 3a^2b^2 + 2ab^3$

Therefore,

$\sqrt{\frac{a^4+b^4+(a+b)^4}{2}} = \sqrt{(a^2 + b^2 + ab)^2} = a^2 + b^2 + ab$

$a^2 = 121$

$b^2 = 10{,}000$

$ab = 1{,}100$

The sum is equal to 11,221. Thus,

$$\sqrt{\frac{11^4 + 100^4 + 111^4}{2}} = 11{,}221$$

The substitution can be hidden and almost invisible as in the following problematic situation.:

Problem 10

Find all whole number values for k, such that $\sqrt{k}$ and $\sqrt{k+45}$, are whole numbers.

Intuitively: For what value of k is $\sqrt{k}$ a whole number? 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, …

Obviously $k = 4$ and $k = 36$ are solutions. Is that all solutions?

If $\sqrt{k}$ and $\sqrt{k+45}$ are whole numbers, we can write the following as a sort of substitution.

$\begin{Bmatrix} \sqrt{k} = n \\ \sqrt{k+45} = m \end{Bmatrix} <=> \begin{Bmatrix} k = n^2 \\ k + 45 = m^2 \end{Bmatrix} => m^2 - n^2 = 45 => (m-n)(m-n) = 1 \cdot 45 = 3 \cdot 15 = 5 \cdot 9.$

This gives us three different solutions, as follows: {$m = 23$, $n = 22$}, {$m = 9$, $n = 6$}, and {$m = 7$, $n = 2$}.

Solutions: $\underline{k = 4 \vee k = 36 \vee k = 484}$.

CONCLUSION

These eight examples are just examples. You could search for other problems that use substitutions in a successful way. Do not hesitate to make your own substitution on any of the problems I have offered in this article. I hope you now see the method of substitution as important as I do after you have read this article.

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