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Effective mathematics learning through APOS theory by dint of cognitive abilities

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ABSTRACT

The paper dwells on the contributions of APOS theory to the development of teaching and learning of mathematics in school. APOS is an acronym for *action*, *process*, *object*, and *schema*. The theory emerges as an extension to constructivism but with a more focused and robust learner-centered approach to the teaching and learning of mathematics. Proponents of the theory believed that learning occurs initially as an *action* or *activity* in learners' cognitive settings, independent of learners' environment, triggered by cognitive coherence, then it is transformed to *process*, where learner now waits for internalization of the earlier *activity*, preparatory to the occurrence of learning. At *object* level, learner now considers what has been learnt earlier to have been fully internalized into mathematical *object(s)*. Lastly, at *schema* level, the *object* learnt is assumed to have been embedded in the learners' *schema*–a cognitive structure formed as a result of accumulated learning experience, and a complete mental image of what has been learnt is said to have been formed. Against the backdrop of this, the paper looks at how this theory had changed the narrative about teaching and learning of mathematics vis-à-vis the bearing of the theory to other cognitive abilities of the learner such as intelligence and creativity. In the end, the paper suggests the application of APOS theory in teaching and learning mathematics at all levels of learning in Nigeria and beyond.

Keywords: APOS theory, learning, mathematics, cognitive abilities

INTRODUCTION

Constructivism as a theory in education had, in recent years, been widely used in mathematics education, and that it recognizes how a learner constructs new understanding of mathematical schemata and knowledge, integrating it with what he/she might have known already while attempting to learn mathematics (Fosnot, 2013). This includes knowledge acquired prior to school enrollment by the learner (Cobern, 2012). Constructivism is at the center of various philosophical traditions, especially epistemology and ontology (Mcphail, 2016). The origin of constructivism as a theory of learning is traceable to the work of Piaget (1972) in his famous theory of cognitive development (Ultanir, 2012; Waite-Stupiansky, 2017). Thus, constructivism in mathematics education originated from epistemology, which as a theory of knowledge, is concerned with the logicality in its categories of knowledge and the justification for acquiring such knowledge. Hence, epistemology focuses on differentiating between the subjective knowledge of a single knower and conventional knowledge to be acquired by all (Ernest, 2013; Palermos, 2014). In constructivism, it is recognized that learner has prior knowledge and experiences, which are often shaped by his/her social and cultural environments (Doolittle, 2014). Therefore, learning is showcased while learners are *constructing* knowledge out of their own experiences arising from the interaction they might have had with the environment. While the behaviorist school of learning may help understand what students are doing from behavioral point of view, mathematics educators need to know what students are thinking mathematically from their cognitive settings, and how to enrich what students are thinking to make it a complete learning pattern (Gourdeau, 2019; Thompson & Calson, 2017).

Furthermore, constructivism is at the core of psychology of mathematics learning. In his famous work, Piaget (1972) focused on how human beings make meaning out of the interaction between their accumulated experiences and their ideas (Ackermann, 2012) in which case, mathematics ideas. His views focused on human development in relation to what is occurring with an individual as distinct from development influenced by other persons within the same environment (Newman & Newman, 2015). However, for Lev Vygotsky's (1896-1934) theory of social constructivism, the emphasis about learning was on the centrality of socio-cultural learning milieu; based on how learners' interactions with agents of that cultural setting such as adults, influential peers, and cognitive tools (such as knowledge, intelligence, and creativity) are acquired and internalized by learners to form mental structures through the zones of proximal development (Eun, 2019). Expanding upon Vygotsky's theory, Jerome Bruner and other educational psychologists developed the important concept of instructional scaffolding (Belland, 2017; Smagorinsky, 2018), whereby the social or informational environment offers supports (or scaffolds) for learning that are gradually withdrawn as they become internalized and as the learning becomes permanently entrenched in learners' cognitive structures (Bonk & King, 2012).

Against the backdrop of this, the emergence of APOS theory provided a different narrative about mathematics learning. Grounded in Piaget's (1972) epistemology and constructivism, its central tenet is "*objects* of learning". To describe *object* of learning as an emerging construct in mathematics learning, Dubinsky et al. (1997) argue that, when learning mathematics, the mental *processes* of learning taking place in learners' cognitive settings are structured to occur initially as an *action* or *activity.* This *action* is considered independent of learners' learning environment. Therefore learners consider it of little significance to the learning *process*. Secondly, it is a *process*–whereby the learner no longer considers the concept as an *activity* but a *process* concurrently happening in his/her mental setting, waiting for internalization prior to the occurrence of learning. Thirdly, learners now consider the concept as an *object* internalized in their mind and fourthly; the concept is assumed to be embedded in the learners' *schema*–a larger and more comprehensive mental picture of the concept formed as a result of the evidence that learning had taken place (Piaget, 1972). Therefore, *object-process* learning as argued in this theory leads to the formation of mathematical structures in learners' cognitive setting, not as a product of interaction with environmental variables like peers or adults. Hence, this informed the choice of APOS theory to explore students' cognitive abilities in problem solving skills in mathematics within learners' cognitive parameters.

APOS THEORY IN TEACHING & LEARNING MATHEMATICS

The peculiarity of mathematics teaching requires a theory different from the ones that have been in use for decades. This would be an evidence for the continuous research and development in the field. Away from the argument of environmental variables or behavioral pattern influential in mathematics teaching and learning, APOS theory emerged to argue differently. To teach using APOS theory, the teacher has to be conscious of learner's mathematical ability because that is what he seeks to explore. On the basis of that exploration, the teacher then applies the provisions of the theory to address the identified learning loopholes in learner. APOS theory provided for four sequential steps required by learners to learn particular mathematical information and teachers too have to be conscious of the steps to enable effective instruction. The steps as reflected in the acronym, APOS are *action* or *activity*, *process*, *object*, and *schema*.

Action or Activity

In mathematics learning, *action* occurs as a cognitive transformation of a mathematical concept in response to what the learner intends to learn–an external stimulus or a mathematical problem, say for example, Muhammad wants to invest N5,000 over 10 year's period. His bank offers him 6.5% p.a. of compound interest. His uncle offers him 7.5% p.a. of simple interest. From this mathematical problem, the following questions could be generated.

1. Which of the investments is the best option for 10 years?
2. How much interest does he earn in each option?

When a learner of mathematics is faced with these types of problems, the solution requires a starting point or a trigger usually facilitated by the teacher or a fellow student with the hope of achieving a predetermined goal in the *process* (Arnon et al., 2014; Makonye, 2017). This trigger is known as "*action*" in APOS theory.

Similarly, Makonye (2017) argues that *action* is the initial level of learning mathematics. In which, making sense of a mathematical situation starts from *action* level. Thus, when an *action* is conceived, it leads to an operation outside of learner's mind for learning to occur. According to Arnon et al. (2014) and Maharaj (2013). *Action* is an externally conceived transformation of *object* or *objects* learnt in reaction to a given stimulus (learning). Thus, *action* must therefore be performed step-wise and in a sequentially arranged order. This helps in proper assimilation and interiorization of the concept to be learnt. Consequently, when learning is about to occur, *action* sets in to provoke it and make *object(s)* assimilation possible (Piaget, 1983). The assimilation of mathematical *objects* allows for internalization of the concepts. Hence, *action* is triggered by external influence to indicate the potential of learning to take place (Arnon et al., 2014). This potential for learning to occur initiates *process* in APOS theory.

Process

Learning of mathematics entails making sense of mathematical constructs in learners' mental structures through some *processes* (Dubinsky, 1991). These *processes* are the pre-requisites for making learning of mathematics successful. Such *processes* include "interiorization, encapsulation, coordination, reversal, de-encapsulation, thematization, and generalization" (Arnon et al., 2014, p. 17). For interiorization, it occurs when individual (learner) carries out an *action* (usually mathematical) in his/her mental settings repeatedly (Maharaj, 2013; Makonye, 2017). According to Maharaj (2013), *process* entails deep reflection of what has earlier been "interiorized into a mental *process*" (p. 4) during learning. Otherwise interiorization cannot be said to have taken place. Therefore, *process* is required to make learning of mathematics attainable. Though, Sfard (1991) argues that *process* encapsulation is difficult to make in *object* formation. However, when *action* is initiated and *process* is started, objectification will set in, ready to occur in learners' cognitive structure.

Object

In APOS theory, *object* occurs when a learner reflects on some mathematical tasks in relation to a particular process. Herein, the learner automatically becomes aware of the *process* in its totality (Mathews & Clark, 2003). The learner then realizes that it's a sort of a transformation through a particular *process* that leads to the development of mathematical *objects*. When this occurs, the learner is said to have an *object* conception of the concept that is to be learnt. Makonye (2017) and Sfard (1991) argue that a mathematical *object* is a structure-like pattern. This structure connotes the metaphor of noun. Therefore, in mathematics, such concepts as equations, expressions, quantifiers, numbers, formulae, and figures, are all mathematical *objects* within the realm of APOS theory.

However, Sfard (1991) considers *object* as a mathematical *entity* existing in the realm of space and time. Herein, Sfard (1991) opines that taking mathematical *object* as an entity means "referring to it as it was a real thing–a static structure". A characteristic Sfard (1991) referred to as "duality" (p. 6); both as concept and as a structure. Meanwhile, these series of transformations are known as encapsulation of a *process* into a mathematical *object* (Dubinsky, 1991). When this happens, individuals (learners) become aware of the entire *process* of learning through transformation (Cottrill et al., 1996). In this sense, *object* conception is said to have occurred and *schema* will now appear to make the APOS assertion complete.

Schema

According to Baker et al. (2000), a *schema* in mathematics, is a constellation of *action*, *process*, and *object* conceptions in mathematics learning. This entails previously constructed *schemas*, which are harnessed to enact mathematical concepts (i.e., structures) that are utilized in problem situations. These *schemas* emanate as a new form of relation between emergent and previous conceptions (involving *action*, *process*, and *object*). Similarly, other *schemas* are formed and re-formed in yet another cycle of *actions* and *processes* (Baker et al., 2000; Makonye, 2017). To this end, learners of mathematics work with variety of mathematical tasks that are related to the concepts to be learnt differently. This is dependent on his or her conception of the task (Trigueros & Martinez-Planell, 2010). The occurrence of this signifies the formation of *schema* in individuals' cognitive structure.

However, in APOS theory, it can be argued that there is sequential progression or shift from *action* to *process* to *objects*. These arrays of *actions*, *processes*, and *objects* are arranged in *schemas*. The *schemas* often appear more like a logical progression from one conception to another (Czarnocha et al., 1999). In mathematics therefore, such concepts as bank loans and interest accrued to which an individual is obliged to be responsive to, as a result of a sealed agreement between him and the financial institution, can be considered a *schema* formed after the remaining *processes* have taken place.

LEARNER COGNITIVE ABILITIES & APOS THEORY

Cognitive abilities play a key role in learning mathematics, which had, over the years received less attention on the part of teachers. The need to focus on learners' cognitive abilities stemmed from the need to help students develop meaningful understanding, reasoning ability, problem solving skills other higher order mathematical abilities including intelligence and creativity (Wardani et al., 2011). Many learners were found to develop weak cognitive abilities arising from the failure of teachers to focus their attention on developing learners' understanding of mathematics (Collins, 2018; Little, 2012). Similarly, it was observed that teachers had little time to observe and engage their students in learning so as to diagnose students' learning difficulties in order to help them in improving their mathematical abilities (Lai & Hwang, 2016). As a consequence of this, teachers were advised to not only try and have better understanding of mathematics but have the ability to select relevant and appropriate teaching strategies/methods to fit mathematics areas deemed difficult and that suit various abilities of learner. Due to this teacher variable concern in learning mathematics, comes the need for applying theories such as APOS in order to address the problem.

APOS theory is cognition-based and seeks to extend such works as Piaget's (1983), done in the past in order to have more robust and learner-centered approach to learning mathematics. The sequence of activities in APOS culminated in learning. However, firstly, in learning a given concept in mathematics, learner has to initiate a cognitive *process* to implement the learning of that concept through an *action*, and then interiorize this *action* into *process*. The emanating *process* is, in turn, encapsulated into *objects*. And finally, learner harmonizes these mentally constructed *objects* into *schema* for a concept to be learnt. As an extension to constructivism, APOS theoretical viewpoint is that the role of the teacher is not to transfer his understanding of a concept to the students verbatim or through rote learning. Instead, the role of the teacher is to create scenarios in which students are likely to create these *actions*, *processes*, *objects*, and *schemas* for themselves without having to be directed to do that. However, this does not mean students should be allowed to discover all, or even most, of the mathematics for themselves. Rather, a teacher carrying out a pedagogical approach based on APOS theory should structure lessons intended to provide students with learning experiences to reflect *actions*, *processes*, and *objects* of mathematics in an attempt to help the students build an elementary mental constructs and organize them into a coherent *schema* of mathematics (Siyepu, 2013).

Therefore, the aim of this paper is to use APOS theory as a base for understanding learner cognitive abilities such as knowledge, intelligence and creativity to provide a theoretical analysis of some mathematical constructs and problems in relation to the problem solving ability of the learners; as well as the application of APOS theory in the enhancing cognitive abilities of the learner.

APOS Theory in Teaching Mathematics

Mathematics teaching requires a strong content and pedagogical base. Teachers have to be well grounded in both content and pedagogy to carry out the task of teaching efficiently (Ball et al., 2008; Shulman, 1987). Only then, can mathematics learning be made to be possible by the learners of mathematics (Darling-Hammond & Bransford, 2007). According to Simon (1995), the need for effectiveness in teaching mathematics triggers some reforms in recent times. The application of constructivists approach to teaching was one of them, as well as curriculum design, and development of teaching materials. According to Simon (1995), since pedagogical approach to teaching is multi-dimensional, teacher, as the core in the *process*, has to be given much attention. However, because of its importance in teacher education, mathematics teaching attracts several studies, each with a particular focus (see Avalos, 2011; Grossman & McDonald, 2008). According to Ball et al. (2001), mathematics teaching has been a subject of concern for decades in which the focus was mostly on teachers and their knowledge of teaching mathematics.

This concern for the need for effective teaching of mathematics was further amplified by the recorded failure and underperformance of learners in various aspects of mathematics in Nigeria and elsewhere around the world (Awofala et al., 2020). This consequently led to the sponsorship of many projects internationally (Atweh et al., 2011) in order to address problems bedeviling the teaching of mathematics (Adler & Venkat, 2014; Askew et al., 2012) as well as enhance the performance of learners. In view of this, the need for developing an alternative to effective teaching of mathematics is paramount. Therefore, applying APOS theory to address these lingering problems of mathematics poor performance by the learners, and the enhancement of pedagogical competence of teachers is desirable. This will help create pedagogical balance between teachers of mathematics and learners as well and ultimately rid the system from the seemingly intractable problems bedeviling teaching and learning transaction in education industry.

Many attempts have been made in the past to devise a means of addressing teaching problems in mathematics. This takes variety of approaches; ranging from developing a suitable teaching strategy (Hill & Ball, 2004; Rowland & Turner, 2007), to the application of a specific teaching approach (Niess, 2005), to the enhancement of learner motivation through appropriate teaching (Tella, 2007), to the usage of appropriate instructional materials in teaching mathematics (Ball, 2003) and a host of other approaches. Few of such efforts prove successful while others were not. The myriad of these problems led to the emergence of many theoretical assumptions regarding teaching mathematics for sound mathematics learning. This ranges from Shulman (1986) who came up with the concept of pedagogical content knowledge, Fennema and Franke (1992) teachers' knowledge and its impact model, Ball et al. (2008) content knowledge for teaching, Rowland (2009) knowledge quartet and lately mathematics for teaching (Adler, 2005). Each of these theories provide a unique approach to teaching mathematics, given the circumstances that led to their emergence and the failure of the preceding theories to provide realistic solution to the problems of teaching mathematics.

Therefore, since its emergence, APOS theory had been used in different studies across the globe and most of these studies were unanimous as to the efficacy of the theory in addressing learner problems especially cognition-related. In most of these studies, when compared with other approaches to teaching mathematics, APOS theory proved viable. Thus, this intended study will use APOS theory in studying mathematics problem solving performance of students in the north-west geopolitical zone of Nigeria.

Mathematical Intelligence vs. Mathematics Learning

Intelligence, as an integral aspect of learners' cognitive ability, has been defined differently: firstly, the ability of learner to cultivate such skills as abstraction, logic, understanding, self-awareness, self-worth, learning by doing, knowledge, conception, reasoning, creativity, critical thinking, and problem-solving skills (Renatovna & Renatovna, 2021) amongst others. More succinctly, it can be described as the ability and/or capacity to perceive or infer information, and to retain it as an acquired knowledge to be applied towards solving some behavioral problems within an environment or a given context of learning in different areas of human endeavor (Binkley et al., 2012; Wang, 2009). In this paper, the concern is that of mathematical intelligence as it concerns learning–ability to retain information and retrieve it when there is need for applying it, which involves learners' cognitive ability and how they use it in learning mathematics.

Mathematical intelligence is the intellectual ability showcased by learners of mathematics, which is observed through complex cognitive achievements and high levels of motivation and self-awareness regarding mathematical problems required to be solved (Tirri & Nokelainen, 2012). Mathematical intelligence enables learners to remember descriptions of mathematical ideas and use those descriptions to solve a given mathematical problem. It is a cognitive *process*, which gives learners the cognitive abilities to learn, form concepts, understand, and reason, including the capacities to recognize patterns, linkages and relationships, innovate, plan, and solve problems, and employ language to communicate flawlessly (Román-González et al., 2017). Studies have shown that many future careers for learners of mathematics at secondary school levels in science, technology, engineering, arts, and mathematics disciplines require exceptional mathematical intelligence to be possible (see Kang, 2019; Wannapiroon & Pimdee, 2022). Statistics and data science–two prominent areas of mathematics with wide-ranging applications, have proven relevant in these areas. Therefore, in order to provide a working template for the policy makers and parents alike, this paper provides an opportunity to explore the possibility of integrating mathematical intelligence or make more pronounced in the curriculum of secondary school.

Mathematical Creativity vs. Mathematics Learning

Creativity can be described as a phenomenon, whereby something new and valuable is formed. Creativity as an area of interest to scholars is found in a number of disciplines, which primarily include but not limited to psychology, business studies, cognitive science as well as mathematics (Hernández-Torrano & Ibrayeva, 2020). There are many forms of creativity most of whom scholars argue have fallen within four factors known as "the four Ps"–*process*, product, person, and place (Runco & Kim, 2020).

Process as an aspect of creativity is shown in cognitive approach that tries to describe thought-*process* and techniques for creative thinking (Ritter & Mostert, 2017). As a *product,* creativity usually appears in attempts to measure creative product (usually psychometrics) and in measuring creative ideas as well (Patel, 2013). Hence, the psychometric approach to creativity reveals that it involves the ability to produce more information that would be expected of a learner when assigned to do a particular learning task (Jiboye et al., 2019). On the nature of creative *person,* creativity is considered more of general intellectual habits, such as openness, levels of ideation, autonomy, expertise, exploratory behavior, and so on (Collard & Looney, 2014). On *place,* creativity is considered within the circumstances in which it flourishes, such as degrees of autonomy and access to resources and information. Creative lifestyles in respect to places are characterized by nonconforming attitudes and behaviors as well as flexibility in dealing with the variables in that particular environment (Gyuse et al., 2014; Patel, 2013).

Research on creativity as a psychological construct had been popularized recently (Akgul & Kahveci, 2016). Domain-specific mathematical creativity came on the radar of scholars as a sub-set to the general creativity (Kozlowski et al., 2019). This is due to transdisciplinary nature of mathematics in a world that is rapid in innovation, technological inventions, scientific breakthroughs, solution-orientated creations and massive production of industry-related and household gadgets (Sriraman & Haavold, 2017). Therefore, the need by educators to focus attention on mathematical creativity is timely. Thus, mathematical creativity has been explained variously by researchers (see Runco, 2010; Torrance, 1967). For instance, mathematical creativity was defined based on four categories/features; fluency, flexibility, originality and elaboration (Akgul & Kahveci, 2016; Lev-Zamir & Lekin, 2011).

According to Reiter-Palmon et al. (2019) and Torrance (1967), *fluency* is a frequency for relevant ideas and shows the learners' capability to produce various different responses when asked a particular learning task. This mathematical creativity component is usually described as the number of relevant responses to a given mathematical problem. It is also linked to the development of thoughts and the procedure for knowledge production (Faizah, 2011; Leikin et al., 2013). *Flexibility* relies on the number of categories in a respondent's ideas or responses, while *originality* is defined as the uniqueness of the solutions of students in response to a particular mathematical problem posed (Lev-Zamir & Leikin, 2011; Reiter-Palmon et al., 2019). Thus, it is considered as a unique approach to creative products (Leikin, 2009; Torrance, 1967). *Elaboration* is the number of details given by the respondent and the thorough explanation of the specific mathematical problem posed (Leikin & Lev, 2013).

Therefore, in mathematics learning, the need for teachers to focus their attention on developing students' mathematical creativity based on their ability to be mathematically fluent, flexible, original and elaborate is key in the success of applying APOS theory to teach mathematical concepts. This study intends to explore learners' mathematical creativity through the application of these four features of fluency, flexibility, originality, and elaboration in learning mathematics using APOS theory as a tool of instruction.

CONCLUSIONS

The paper delved much on the suitability of APOS theory in teaching and learning mathematics. As an extension to constructivism, APOS theory came with certain provisions that can be considered more robust and learner-orientated than the previous constructivists' theories. Cognitive abilities of learners such knowledge, intelligence and creativity were looked into as the determinants of success in learning mathematics especially at secondary school level. The paper argues that to address the loopholes in learning occasioned by cognitive-deficiencies, the use of APOS theory method of instruction can help in mitigating the problems of mathematics learning and ultimately enhance learners' mathematical knowledge, intelligence and creativity.

Suggestions

1. Teachers are advised to apply APOS theories in their teaching because it helps them identify the needs of their learners and the level at which learners operate based on the provision of the theory.
2. In line with the provision of APOS theory, teachers should observe the manifestation of mathematical intelligence in their learners so as to help them in nurturing that intelligence in order to prepare them for future careers in various STEM-related disciplines that require some level of intellectual capacity to excel.
3. Teachers should also watch their students' mathematical creativity through the lens of APOS theory. Features required to exhibit mathematical creativity, i.e., fluency, flexibility, originality and elaboration should be watched closely by teachers as they teach their students.

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