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Enhancing students mathematical thinking using applets

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ABSTRACT

This article presents two different classroom situations in which two pairs of students, one in grade 2 and the other in grade 6, attempted mathematically challenging problems through GeoGebra applets. Their reasoning, conjecture making, and argumentation is analyzed using Pea's (1985, 1987) theory of technology as amplifier and reorganizer and the notion of internal and external representations. GeoGebra played the role of amplifier and reorganizer in enabling students' explorations and led them to make conjectures. Furthermore, it was observed that students' static internal representations were enhanced to more dynamic external representations and this played a significant role in developing students' thinking. The three aspects of fidelity have been discussed with regard to the use of GeoGebra based applets and how such applets need to be critically designed and assessed in order to support students' learning.

Keywords: amplifier, applets, fidelity, mathematical thinking, reorganizer

INTRODUCTION

The role of digital tools in learning mathematics has intrigued mathematics education researchers for many years. Many researchers have focused on the use of dynamic geometry environments (DGEs) in mathematics learning (Falcade et al., 2007; Herbst, 2004, 2005, 2006; Laborde et al., 2006; Lingefjärd & Ghosh, 2016; Moreno-Armella et al., 2008).

Purpose of the Study

This study was designed to explore the ways in which DGEs, such as GeoGebra, offer opportunities for students to explore geometrical objects and their properties and to reason mathematically about them. One consideration of the study was whether students in compulsory school with no prior experience on a DGE, can engage with problem solving in a meaningful way. The students had no prior exposure to GeoGebra; however, they had attempted geometrical tasks using manipulatives in paper-pencil mode in their regular classes. In this study, students' approach to solving the problems and the reasoning used by them was of primary interest. Another important aspect of the study was to explore the nature and characteristics of tasks, created as dynamic geometry applets to elicit students' mathematical thinking. The applets used in this study were constructed within the *Matematiklyftet* (lifting the mathematics), a national in-service project for mathematics education in Sweden. In 2011, the mathematics curriculum of compulsory school in Sweden underwent a major revision in which five different competencies namely–communication, conceptual understanding, problem solving, procedures, and reasoning were given special emphasis. Researchers participating in the study, developed different mathematical tasks to enhance and elicit students' mathematical thinking and communication skills. Among them, the authors of this article developed GeoGebra based applets for grade 2 and grade 6 students.

The purpose of designing mathematical challenges in the form of GeoGebra applets was primarily to make the tasks accessible across a wide range of schools. It was intended that primary and middle school students be represented in the project. Thus, students from grade 2 and grade 6 were chosen to be participants in the study. Grade 2 students were 7 and 8 years old whereas the grade 6 students were about 11 years in age. There were only two GeoGebra applets used in this study, one for grade 2 students and one for grade 6 students. In the task presented to grade 2 students (as shown in **Figure 1**), the students were required to move the shapes placed on the right, into the blue square frame to fill the inner square. The shapes are moved by dragging but their orientation could not be changed or flipped. We also show a screen shot of the task attempted by grade 6 students (see **Figure 2**). A larger rectangle is divided into four smaller rectangles by two perpendicular lines. These two lines meet at a point (marked in blue in the applet). Dragging this point inside the larger rectangle can be used to modify the areas of the smaller rectangles. The students have to identify the position of the blue point so that the purple and blue rectangles have equal area.

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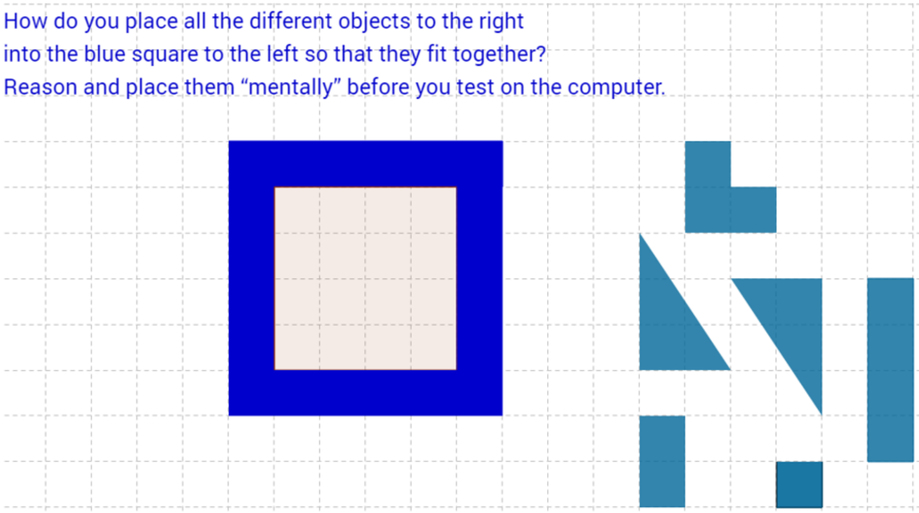

Figure 1. An applet by GeoGebra at address https://www.geogebra.org/m/xaXVBXcE

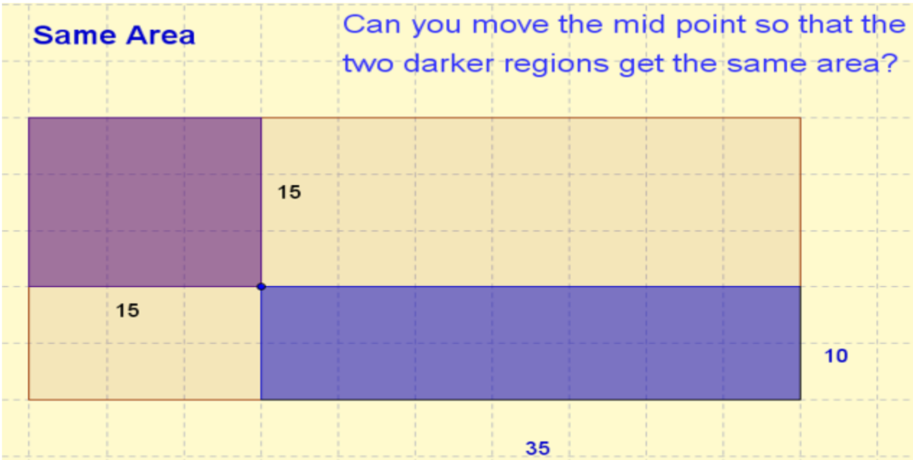

Figure 2. An applet by GeoGebra at address https://www.geogebra.org/m/uA3XhTb2

These tasks were administered to grade 2 and grade 6 students across a few schools in Sweden. These schools volunteered to let their teachers use the tasks for their students. However, only two schools allowed the filming of the students solving the tasks. The geometrical tasks were assigned to students in pairs since this would encourage discussion and collaboration between peers. The researchers identified the two videos (one of a grade 2 student pair and one of a grade 6 student pair), as exemplars as they were rich in terms of student's reasoning, approach to problem solving and in terms of audio quality. Also, the students in the selected videos had completed the task assigned to them.

We first describe the learning theories, the theoretical framework of our study, and the research questions. We then discuss the methodology of the study and we have elaborated on the method of data analysis using the lens of the various theoretical frameworks we described. After that we discuss the findings of the study and we explain the limitations and pitfalls. Finally, we summarize the conclusions of the study and point to further research on the theme of using DGEs for enhancing students' mathematical thinking.

THEORETICAL FRAMEWORK

This section discusses the theoretical frameworks, which characterize students' mathematical thinking and reasoning through the use of dynamic geometry applets.

Learning Geometry Through Internal and External Representations

There is a connection between the way we develop our understanding of mathematical objects and our internal concepts related to them. Learning occurs as we notice *external representations* of mathematical objects and then construct our own *internal representations* of the same objects. Internal representations can be formed from concrete objects or from verbal clues by an adult or peer. Showing a picture of a square may help a child develop an internal representation of the concept of 'square'. Describing a square, as having 'four equal sides' is a verbal cue, which may lead to forming an internal representation. To form a holistic internal representation of a square, one needs to experience it in different forms. Dienes (1963) propounded the benefits of presenting different representations of mathematical objects:

> The *perceptual variability principle*: To abstract a mathematical concept effectively one must meet it in different situations to perceive its purely structural properties.

> The *mathematical variability principle*: As every mathematical concept involves variables, all variables need to be varied if the full generality of the mathematical concept is to be achieved.

Internal representations comprise of ideas or mental images, which we refer to mentally. They help us communicate ideas about mathematical objects and concepts, even abstract ones. The notion of internal and external representations was used by Vygotsky (1978) who referred to them as external tools and internal signs, and about the mental transformation between the external and internal. According to Falcade et al. (2007),

> but the link between tools (externally oriented) and signs (internally oriented) goes beyond pure analogy in their functioning and rests on the real tie that can be recognized between particular tools and particular signs. One could say that externally oriented tools may be transformed into internally oriented tools (p. 321).

The cognitive process of going from an external to an internal representation of the verbal material is called a *verbal representational connection*, (see Mayer & Sims, 1994). We describe our thoughts to another person using a visual explanation. For example, we say "a parallelogram is a geometrical figure in which opposite sides and opposite angles are equal" and draw examples on paper. This will enable the other person to construct a mental representation of the visually presented system.

The cognitive process of going from an external to an internal representation of visual information is called a *visual representational connection*. We see square shaped objects in the real world and we experience the idea of a square. These squares are not as perfect as the one we have in our mind but nevertheless we call them squares.

Finally, we need to construct referential connections between the two mental representations (verbal and visual), that is, the mapping of a *structural relation* between the two representations. Building a referential connection involves noting that a statement such as "a triangle has three vertices" is analogous to a static image. Vygotsky (1978) referred to this process as internalization as described by Falcade et al. (2007):

> In this respect, a specific tool may function as a semiotic mediator. At first, externally oriented, a tool is used in action to accomplish a specific task, then, within semiotic activities under the guidance of an expert (for instance, the teacher), the articulation of new signs, generated by (derived from) actions with the tool, may foster an internalization process producing a new psychological tool. This new tool is internally oriented, completely transformed, but still maintains some aspects of its origin (p. 321).

Our internal representations enable us to identify, recognize or interpret the different external representations encountered by us. External representations are physically embodied observable configurations, interpreted by us as belonging to structured systems and their representing relationships. As we build connections between our internal representations, we learn to use them and use them more flexibly. Developing this flexibility between different forms of representation is an important aspect of learning mathematics. In later sections of this article we shall see how working on the GeoGebra applet based tasks elicited student's external as well as internal representations.

Visualization: A Key Aspect in Developing Geometrical Thinking

Visualization is another process in the understanding and construction of mathematical concepts. This is especially true in the case of geometrical concepts. A DGE facilitates visualization, transforms the possibilities for representation, and also has an impact on the conceptualization of mathematical objects and internalizing their meanings (Falcade et al., 2007; Moreno-Armella et al., 2007). The contribution of technology in teaching and learning of mathematics is perceived as strongly linked with dynamical interactive graphical representations (Laborde et al., 2006).

In a DGE geometrical figures and shapes can be reorganized using dragging features and this provides a dynamic opportunity to the learning of geometry. It allows the user to perform investigations and thus affords the possibility of a dynamic visual representation of geometry concepts in a physical sense. Such investigatory activities are hard to experience in a static environment such as paper and pencil (González & Herbst, 2009). In later sections we shall see how students of grade 2 and grade 6 worked with geometrical shapes via GeoGebra applets and made conjectures about them.

DGE as Amplifier and Reorganizer

The choice of mathematical tasks enabled by technology has important implications for students' quality of mathematical thinking. One of the important roles that digital technology can play in the mathematics classroom is that of a cognitive tool. Pea (1987) defined cognitive technologies as those that transcend the limitations of the mind (e.g. attention to goals, short-term memory span) in thinking, learning and problem solving activities (p. 91). Further, Pea (1987) distinguished between two aspects of cognitive technologies. He claimed that technology can be an *amplifier* and/or a *reorganizer* of mental activity (Pea, 1985, 1987). These two metaphors have been widely applied in mathematics education. The term *amplifier* refers to the fact that technology performs tedious computations quickly and thus students can focus on making observations and developing insight rather than be caught up by manual procedures. According to Sherman (2014):

> When technology is used as an amplifier, it performs more efficiently tedious processes that might be done by hand, such as computations and generation of standard mathematical representations.

For example, while using a DGS for drawing triangles and measuring angles or side lengths, students can quickly produce and observe many triangles. In this process the tool does not change student's thinking as they are still focusing on angle measurements or side lengths. They could have done the same task on paper and pencil using ruler and protractors. However, the DGS makes the process more efficient by allowing the student to drag a vertex of the triangle and produce many triangles in one continuous motion.

However, when technology is used as a *reorganizer*, it extends students' thinking by giving them access to higher level processes. These include looking for patterns, identifying invariances or making and testing conjectures. Thus, if a student uses a DGS to drag a vertex of a triangle to observe the invariance of the interior angle sum (of 180 degrees) amidst variations of the individual values of the angles, she is using DGS as a reorganizer. According to Sherman (2014):

> As a reorganizer, technology has the power to affect or shift the focus of students' mathematical thinking or activity … Students might use a DGS to construct a triangle and its medians in order to look for patterns, make and test conjectures about the relationships between the medians of a triangle.

Pea's (1987) theory has been used in several contexts to address various aspects of mathematics education. Technology can be used to reorganize the curriculum or reorganize the structure of the learning environment. However, the metaphor of reorganizer can also be used in the context of specific mathematical tasks, in terms of how technology can influence what students do and how they think mathematically while engaging with those tasks. It is this aspect of the reorganizer metaphor, which we shall focus on in this study. In short, we shall see how the GeoGebra applets function as amplifiers and reorganizers in developing student's mathematical thinking

The Three Aspects of Fidelity: Mathematical, Pedagogical, and Cognitive Fidelity of DGE-Based Representations

While using technology for exploring mathematical concepts and problems, it is relevant to assess its pedagogical, mathematical and cognitive fidelity. Zbiek et al. (2007) describe mathematical fidelity as

> "faithfulness of the tool in reflecting the mathematical properties, conventions, and behaviors (as would be understood or expected by the mathematical community)" (p. 1173).

Let us consider the function $f(x)=(x^2-1)/(x-1)$. If a graphics calculator graphs this function, it may produce the linear equation $y=x+1$. This is inaccurate as the function $f(x)$ not is defined at $x=1$ and the correct graph of $f(x)$ should have a point break at $x=1$. Thus, the mathematical fidelity of the tool is compromised in relation to graphing of the function.

Zbeik et al. (2007) describe cognitive fidelity as

> "the faithfulness of the tool in reflecting the learner's thought processes or strategic choices while engaged in mathematical activity" (p. 1173).

A tool has cognitive fidelity if the produced external representations match the user's internal representations and enhance their conceptual understanding. If appropriately used, a DGE has good cognitive fidelity.

The third kind of fidelity is that of pedagogical fidelity, which, according to Zbeik et al. (2007), is

> "the extent to which teachers (as well as students) believe that a tool allows students to act mathematically in ways that correspond to the nature of mathematical learning that underlies a teacher's practice" (p. 1187).

Pedagogical fidelity refers to the tool's ability to support students' explorations and learning. In a DGE, the dragging feature can afford this kind of fidelity. We can use sliders to vary the values of 'a' and 'b' in the equation $y=ax+b$ and observe the change in the graphical representations of the linear equation. The level and degree of the types of fidelity vary among technology tools and should be considered while selecting and evaluating appropriate tools for students' explorations. The aspects of fidelity must also be kept in mind while designing exploratory tasks for students.

The theoretical frameworks described above may be classified into two categories. Internal and external representations focus on understanding students' thinking and reasoning in a DGE environment, while Pea's (1985, 1987) theory of amplifier and reorganizer and the three aspects of fidelity are related to the design and positioning of DGE based geometrical applets for mathematical problem solving. Both aspects are important for framing the research questions of this study. Among the following three research questions, research question 1 is related to developing students' reasoning and conjecture making skills using a DGE while research questions 2 and 3 are related to design of tasks in a DGE.

Research Questions

The objective of the study was to address the following research questions:

1. What is the nature of students' internal and external representations while engaging with tasks in a DGE?
2. Do tasks design in the form of DGE applets function as amplifiers or reorganizers (or both), in enhancing student's geometrical thinking?
3. Do DGE representations satisfy the mathematical, cognitive and pedagogical fidelity criteria?

METHODOLOGY

The study is qualitative in nature. The video recordings of two pairs of students were analyzed to address the research questions using the theoretical frameworks mentioned before. Students' conversations in the video and their attempts to solve the problems posed via GeoGebra applets were the primary source of data.

Participant's Background

The video recordings of two grade 2 students from a small elementary school and two grade 6 students from a middle school in Sweden were identified from among 6 video recordings which were filmed as a part of the *Matematiklyftet* project. The students who were filmed had no prior experience in working with a DGE. The students in grade 2 were familiar with geometrical shapes (such as rectangles, squares, triangles and circles part of the curriculum) but were not familiar with the relationships between them or with their properties. The students in grade 6 were familiar with algebraic manipulations but were not familiar with the concept of a function. The video clips of the two pairs of students were selected as exemplars, mainly because of the richness of their responses and also because these pairs had completed the tasks assigned to them. By richness of response, we refer to the interaction between the two students within the pair as well as the fact that the students had articulated their reasoning whenever prompted. A teacher is present in the video clip of the grade 2 students. Her role was to introduce the task and ensure that they clarify their doubts with regard to the task if needed. Further the teacher also prompted a few questions to elicit the students' reasoning. In the video clip of the grade 6 students no teacher is present. A teacher was present in the vicinity, to address any queries related to the task, but the students managed to work through the task on their own. One of the objectives of the study was that the students should communicate their thinking with minimum intervention and hence the teacher's presence was not considered mandatory.

The video films analyzed in this study may be found at the following links

Grade 2: https://youtu.be/QK8WxZH6NFQ

Grade 6: https://youtu.be/J3sxLVS3lUE

The Intervention

Explorations by grade 2 students

The video shows two students in grade 2, Judy and Matteo, sitting at a table in the classroom together with their teacher, Anna. They are sitting in front of a computer and the applet shown in **Figure 1** is visible on their screen. The text in the applet was originally in Swedish but has been translated to English for the purpose of this article.

The conversation between the two students, as they explored the task, is translated into English and it starts with the teacher explaining the task to the students:

> 0:05 T: "Now you are going to try to work with GeoGebra on the computer."
>
> 0:10 T: "The task is to move the objects on the right into the blue frame on the left."

The teacher is communicating the problem to the students and they are listening while going through organization and recognition (which is evident from their facial expressions).

> 0:15 T: "I want you to think first and agree together on the way to do this, in terms of where to place the objects."
>
> 0:25 T: "I want you to use the correct mathematical names for the objects when you are reasoning, so that you understand each other's intentions and thinking. Do you want to try?"

The students have been listening silently with some interest. The teacher invites them into the conversation. The following excerpts from their conversation will reveal how the students traverse through the phases of organization, recognition, and representation.

> 0:30 T: "Tell me what you are thinking."
>
> 0:35 Judy: "We could drag this one over to here (she points with a finger at the monitor)."

She is communicating with both words and body movements.

> 0:37 T: "What should we call that object?"
>
> 0:39 Judy: "A triangle."
>
> 0:41 T: "Where do you want to place it?"
>
> 0:42 Judy: "In the corner." (She is referring to the bottom left corner of the blue frame)
>
> 0:44 T: "Matteo, what is your opinion about this?"

0:45 Matteo: "I think we can place the other triangle next to the first triangle. That will create a standing rectangle."

Matteo's conclusion regarding the 'standing rectangle' is based on the idea that two congruent right-angled triangles, when joined on their hypotenuse, form a rectangle. This notion seems to be part of his internal representation.

0:54 T: "All right…"

0:56 Judy: "We can put the long rectangle into the middle. She is pointing with her hand.

0:57 Matteo: "We put one triangle up a little bit to the right… Then we place the second triangle next to it."

1:05 Matteo: "That will create a standing rectangle in the upper left corner."

1:12 T: "Seems good and how do you think we should succeed, Judy?"

1:16 Judy: "We could move this object looking as an L" ...

She points at it.

1:20 T: "Where to?"

1:22 Judy: "In the corner"

1:24 T: "What should we call that corner?"

1:26 Matteo: "The lower corner?"

1:29 Judy: "And we can add the small square on top of it… Then it will be a square."

1:34 T: "All right and how do you want to continue now, Matteo?"

1:37 Matteo: "We can take the long rectangle to the right…"

1:45 T: "All right, what have we left now?"

1:47 Matteo: "The short rectangle."

1:49 T: "And where should you place that?"

1:51 Judy: "It will fit into the corner; I think…"

1:55 T: "Maybe it will? Why don't you try to do it on the computer now?"

They test if their internal representations and mutual communication have found a way to fit the objects into the square. In their exploration the students realize that only by dragging can they shift the shapes on the right of the screen. The shapes cannot be rotated or flipped nor can their orientation be changed. The task of filling the blue square frame with these shapes has to be completed only via translation. This realization is important for arriving at a solution to the problem.

The teacher supports and encourages the students as they move the objects as in their plan. It takes about one minute and then the teacher says:

2:55 T: "Did it turn out the way you thought it should?"

3:00 Matteo: "No…"

3:05 T: "What did not turn out the way you thought it should?"

Student's approach, of forming a rectangle by joining the two right-angled triangles, placing it in the corner of the square and then filling the other shapes around it, leads to the solution of the task. After this, the teacher changes direction in the conversation and encourages them to reason and find other possible solutions. While trying, the students manage to find an alternate solution to the task. This caught the students by surprise and led to the realization that there could be multiple approaches to solving a problem.

Explorations by grade 6 students

The video shows two grade 6 students facing a computer with the view of the applet shown in **Figure 2**. The text in the applet is originally in Swedish but has been translated to English for the purpose of this article.

The conversation starts with the two students trying to understand the problem.

0:05 Student 1: "No – they are not the same!"

The students are mentally calculating the area of the two darker rectangular regions as the side lengths of the rectangles are displayed on the applet. At the same time, they are communicating fast.

0:15 Student 2: "If we move the midpoint downwards?"

He moves the blue point (referring to it as the "midpoint") with the mouse.

0:25 Student 1: "Now we have 14 times 12 which is 168 and 16 times 8… It's close but not the same!"

0:28 Student 2: "We have to move it up again. We have 16 times 10 and 10 times 14, which is close but not the same. It is not going to be so much in the middle–am I right?"

0:35 Student 1: "Let us go a little bit further to the corners, could we?"

The students move the "midpoint" back and forth for some seconds, and suddenly:

0:44 Student 2: "There–what do we have here?"

0:46 Student 2: "We have 4 times 26 and 16 times 4 so let's move the point to the left a little bit."

The students are computing the multiplications mentally while searching for the right position of the midpoint.

0:52 Student 1: "Now we have 4 times 24 and 16 times 6–yes, we have another solution here! We have found another one!" (there is astonishment in his voice).

The students leave the possibility of dragging the "midpoint" for time being and try to find a pattern for the solutions. When doing so, they leave the external representation of the problem in GeoGebra and instead use the external representation of paper and pencil.

1:30 Student 2: "It will be easier to find a pattern here, maybe a pattern depending on intervals of 6?"

They discuss in a very fast Swedish the multiplicity of 6 and other possible factors. Nevertheless, they agree that they have found a new solution and abruptly they leave the paper and pencil methodology and revert back to the GeoGebra applet to test their method.

2:05 Student 1: "It should be possible to find a solution at 18 times 8 and 12 times 12–yes."

2:10 Student 2: "Maybe there are only two solutions?"

2:25 Student 1: "I think we can place more solutions on the paper like this."

2:30 Student 2: "There must be solutions here and here and here too."

Student 1 sketches multiple points in his paper-pencil diagram. He points to an imaginary line, a symmetrical (but invisible diagonal) on the paper. This is the first argument for a generalization.

2:50 Student 1: "I believe that, theoretically, there should be infinitely many solutions along here."

This is an interesting remark since students at this grade not have met the concept of infinity in mathematics. Student 1 puts a ruler along the points and seems to consider the possibility of generalization.

2:55 Student 1: "Here is a solution, and here is another solution, and here is a third solution."

3:25 Student 1: "The line is not working; only the integer solutions (points) we have. Let us write down what we have. We have that the blue area is y times $(20-x)$."

Student 1 then writes the area of the upper purple rectangles as $x\cdot(30-y)$. The width of the rectangle is taken as 'x' and the height is $(30-y)$. The expression $(20-x)\cdot y$ is representing the area of the lower blue rectangle (**Figure 3**). The students' approach is to equate the expressions for the areas of the two darker rectangles. After some algebraic manipulation they are able to express 'y' in terms of 'x' as the linear equation $y=(3/2)\cdot x$. This is truly a high point of the exploration as they attempt to generalize the problem and arrive at an algebraic proof.

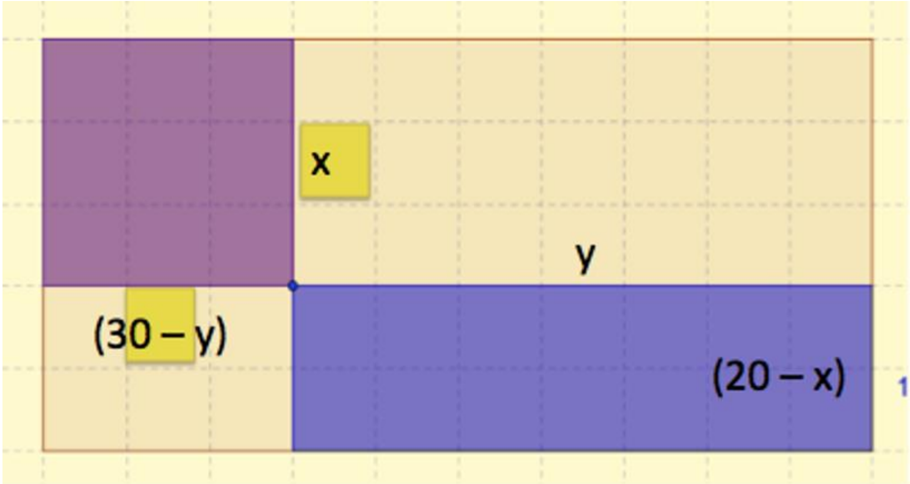

Figure 3. Students use algebra to find the solution of the problem

The applet displays the lengths of the sides of the darker rectangles, which helped the students to mentally calculate their respective areas. Dragging the blue point (which students refer to as the “midpoint”) enabled them to see the change in the areas of the rectangles and this helped them to find positions of the midpoint, which would lead to equal areas of the darker rectangles. The students moved back and forth between the applet and their paper-pencil representation while attempting the task. They used the applet to visualize different areas of the rectangles, identify a pattern and make a conjecture regarding the position of midpoint (leading to equal areas of the darker rectangles), but resorted to algebraic manipulations on paper to generalize their observations in symbolic terms and proving their conjecture. After arriving at the equation $y=(3/2)\cdot x$ the students concluded that for the two rectangles to be equal area, the length ‘*y*’ of the rectangle in the bottom right corner should be one and half times of the width ‘*x*’ of the rectangle at the top left corner.

DATA ANALYSIS

In this section we shall address the research questions of this study. In particular, we shall try to analyze the responses of the students to the GeoGebra applet tasks, which form the primary data of the study, using the lens of the theoretical frameworks, described before. We will highlight the role played by GeoGebra in enhancing students’ internal representations and external representations. We will furthermore deliberate on the dual role of GeoGebra as an amplifier and reorganizer. We will also assess the mathematical, cognitive, and pedagogical fidelity of GeoGebra as a digital tool.

Students’ Internal and External Representations

Students’ communications in the form of reasoning and explanations while attempting the geometry tasks on the GeoGebra applets suggest that the dynamic nature of the tasks offered opportunities different from a static paper-pencil approach. Prior to the study, students’ external representations were largely dependent on verbal explanations and by drawing diagrams in on paper. In the episode with the two grade 2 students, Judy and Matteo, communication and reasoning till 1:55 was related to their organizing, recognizing and forming internal representations related to the objects to the right of the screen. Their internal representations are related to recognition of the shapes and their intuitive understanding about the possible positions of these shapes inside the blue frame. These internal representations gave way to arguments based on reasoning and justification and may be considered as external representations. Matteo (0.57) expressed clear arguments when he expressed his mental animation that the two right triangles can be joined on the hypotenuse to form a rectangle.

From 0:45 onward, Judy and Matteo communicate their reasoning to each other. Even without any form of concrete external representations at hand, Matteo (0:45) and Judy (0:56) are able to see that the two triangles will form a standing rectangle. They seem to be able to objectify the two triangles, and view them as a new representation, that is, a standing rectangle. These first two minutes of the episode highlight the students’ use of internal representations to organize the filling of the blue square frame to the left. Their way of communicating one sentence at the time also enables them to see the solution building up as an internal representation. The teacher’s approach of probing and questioning, encouraging the students to share their ideas, is also very fruitful and supportive to their learning experience (see Fello & Paquette, 2009; Kastberg et al., 2009, Stokero & Van Zoest, 2011).

At this point it will be appropriate to emphasize on the quote from Mesquita (1998), where she argues that:

> figurative representation of geometrical objects may give support to geometrical intuition, which in some situations can be very powerful. Its power comes from the fact that it helps individuals to apprehend relationships among geometrical objects (p. 184).

Judy and Matteo both had internal representation related to properties of geometrical shapes. In the presence of the GeoGebra applet they only needed a few seconds to understand the situation and to be able to apply their understanding of the shapes to the problem at hand. One may argue that the GeoGebra applet is just a virtual representation of the problem and the students using a physical manipulative such as pattern blocks or tangrams could as well have explored it. A physical manipulative (depending on its design) might have allowed the students to flip the shapes and change their orientations leading to various solutions. If the grade 2 students were presented with the same problem via concrete manipulatives, they would have come up with different solutions. The given applet-based task, is a simpler version where the students had to fit the shapes using translations only. However, dragging helped them to place the given shapes into the square frame in different ways and therefore led them to look for multiple solutions.

At the beginning of the mathematical exploration episode with the two grade 6 students, it appears as they visualized that if the “midpoint” is moved to the center of the rectangle, then the two rectangles will have equal area. This intuitive understanding is a part of their internal representation. Since the applet presents a rectangular grid without the center of the grid marked in it, it is not obvious for the students to identify the position of the midpoint, which will lead to the regions having equal area. Dragging the midpoint enabled them to experience the variation of the darker rectangular areas, as they were able to mentally calculate the areas using the dimensions displayed on the applet. It seems as if they conjectured this using internal representations and as they have imagined the solution internally in their minds.

After 2:25, the students develop further confidence and manifest great efficiency in using their external representations as a combination of GeoGebra applet on the screen together with a paper and pencil approach to arrive at a general solution. Student 2 draws the rectangle on a piece of paper, marks the solution points while commenting: “Theoretically I believe that there must be more solutions right here and here.” Once again, he seems to be using his internal representation and expressing it on paper. He marks the points and uses symmetry within the rectangle to find more solutions, guided by the first solution.

Further, the two students discuss the possibility of using the ruler to draw a segment in the rectangle from the bottom left corner to the top right corner so as to illustrate the possibility of infinitely many solutions on this line segment. This is the high point of the episode and illustrates a ‘leap’ in their thinking about the problem as they attempt to find a more general solution. Towards the later part of the episode, the students appear to move back and forth between the external representations of the GeoGebra to external representations on paper and pencil. They use the applet to visualize the areas of the rectangles and observe a pattern, but resort to algebraic manipulation on paper in order to symbolically represent their conjecture and arrive at a proof. Toggling between the two modes of representation led the students from observations to making a conjecture and finally arriving at an algebraic proof.

Later, when the areas of the rectangular regions appear in decimals, the grade 6 students find it hard to deal with in GeoGebra, since the segment from one corner to the opposite is not in the applet. In this situation, they turn back to integer solutions and at 3.25 they resort to using algebra. They leave geometrical interpretations and decide to use algebra instead to find a general solution. They set up algebra notations (after 3.25 one of the students say*: Let us write down what we have. We have that the blue area is y times (20-x)*). The steps written by them on paper are, as follows:

$(20-x)\cdot y=x\cdot(30-y)$ [They have expressed the areas of the blue and purple rectangles using the variables x and y and have equated them]

$20y-xy=30x-xy$ [They have used the distributive property to open the brackets]

$20\cdot y=30\cdot x$ [They see that the terms $-xy$ and $-xy$ can be cancelled from both sides]

$y=(3/2)x$ [y is written in terms of x as a linear equation]

This revelation, that the line $y=(3/2)x$, a diagonal of the larger rectangle, presenting an infinite number of solutions to the problem, is a sophisticated generalization. This line was not explicitly drawn by the students in the GeoGebra applet. The ability to drag the midpoint to create equal areas of the two darker rectangles, led students to make the conjecture that any point on this diagonal could be a possible solution. In fact, one may say that the conjecture emerged from the students’ internal representations but were explicitly proved using external representations comprising a combination of GeoGebra applet and paper-pencil algebra. Finally, proving their conjecture using algebra supports the idea that a DGS cannot itself create proofs but motivates the desire for proof (King & Schattschneider, 1997).

DGE as Amplifier and Reorganizer

The dragging feature of a DGE enables students to explore, experiment, observe variations and invariances, verify permanence or lack of permanence of mathematical properties, leading to conjecture making more easily than in other digital environments where dragging is not possible. Such dynamism is missing in the more traditional paper and pencil environment. In a DGE learning environment, students can construct complex figures, explore a range of transformations on those figures, and witness multiple examples in a short span of time. Such an affordance can hardly be matched by non-computational or static computational environments (Marrades & Gutiérrez, 2000, p. 96). In this study, the grade 2 students used the dragging feature of the GeoGebra applet to try various possibilities of positioning the geometrical shapes inside the big square. This enabled them to explore different solutions very quickly, a fact which supports Pea’s (1987) metaphor of technology as an amplifier. A DGE like GeoGebra can provide students access to multiple solutions in a short span of time. However, after positioning the shapes inside the big square they began to reason and justify their solution referring to the properties of the shapes and their relative positions. For example, they argued that the two right-angled triangles could be joined together on the hypotenuse to form a “standing rectangle”. By doing so, they arrived at a relationship between right triangles and rectangles, a higher-level concept, much beyond the level of grade 2. They further reasoned that this rectangle can be placed in a corner of the blue frame and other shapes can then be arranged around it to arrive at a solution. Later they also realized that there could be multiple arrangements of the shapes within the blue square frame and thus multiple solutions. Here the mathematical task, posed via the GeoGebra applet, played the role of reorganizer as it shifted the focus of students' thinking from merely dragging the shapes and trying different arrangements to reasoning and identifying the arrangements that would lead to a solution.

In the case of the grade 6 students, we also see evidence of DGE acting as an amplifier. Dragging the blue point inside the rectangle enabled the students to quickly vary the two darker rectangular regions and experience several examples, of the darker rectangles with different areas. Thus, the dragging feature led them to visualize their 'mental animation' very quickly and efficiently. As an amplifier, the applet enabled the students to compare the areas of the darker rectangles. However, later in the episode, students began to look for a pattern, namely a search for positions of the blue mid-point, which would lead the darker rectangles to have equal areas. This led them to conjecture that the darker rectangles would have equal area only when the blue 'mid-point' occupies specific positions, that is, along the diagonal of the bigger rectangle. In this process, the applet played the role of a reorganizer. Here the focus of students' thinking shifted from the numerical areas of the darker rectangles to finding a pattern and a general solution. Further, the applet motivated the students to verify their conjecture using algebra and in doing so the it provided them access to a higher level concept, that is, finding the equation of the line as a symbolic representation of their general solution. Hence we found ample evidence of the GeoGebra applets playing the role of amplifier and reorganizer in the two tasks with grade 2 and grade 6 students.

The Fidelity Aspects of DGE

The applets presented to the grade 2 and grade 6 students were interactive in nature and were accurate representations of the actual problems along with their mathematical properties. In this regard both applets may be considered to have a high level of mathematical fidelity. The applet of the grade 2 problem may be considered as a virtual manipulative which has some of the properties of a physical manipulative version. This applet had the constraint that the shapes could only be moved via translation (could not be flipped over or rotated) and that led the students to explore the problem differently. The grade 6 applet permitted dragging of the midpoint to experience variation of the areas of the darker rectangles. In doing so, the applet preserved the actual properties of the original problem and adhered to the criteria of mathematical fidelity.

The applets also had a degree of cognitive fidelity, since they assisted the learners' thought processes while they engaged in the mathematical task. The grade 2 applet allowed the students to drag and move the shapes by translation and to test different solutions. In the grade 6 applet, dragging the midpoint helped to identify patterns and look for positions which led the rectangles to have equal area. The applet, with its mathematical and cognitive fidelity, created an environment, which led to a "leap" in students' thinking and enabled them to make a conjecture. The intuitive conviction that placing the midpoint on the diagonal of the larger rectangle would result in the darker rectangles having equal area, further led to their finding a proof using algebra. Finally, the applets also had a high level of pedagogical fidelity as they helped the students to further their exploration and learning. The applets encouraged the participation of the students, did not require any prior training and the students used them easily. They were able to explore the geometrical aspects of the problem with almost no help from the teacher. The fidelity of technological tools is an important criterion while evaluating their use in the classroom for mathematical exploration and learning.

FINDINGS AND DISCUSSION

In this article we have analyzed elementary school students' explorations of geometrical problems via GeoGebra applets using theoretical frameworks described before. The two examples of student´s exploration described in this article serve as an illustration of how applets designed using a digital tool such as GeoGebra can foster students' mathematical thinking and problem-solving skills. Further, it emphasizes that such applets must accurately represent the geometrical problems for which they have been designed and thus adhere to mathematical, cognitive and pedagogical fidelity aspects. As the students worked in pairs, the peer collaboration enabled them to practice both communication and reasoning. Here we shall summarize how the research questions of the study have been addressed.

Firstly, we have highlighted the role of GeoGebra in developing students' internal and external representations (research question 1). The student's experience in geometry, prior to this study, was mainly related to paper and pencil tasks or compass and ruler constructions and their external representations were focused on drawings and constructions on paper. They had no exposure to geometrical figures in a computer based or DGE environment. Overall, the grade 2 students had very limited classroom exposure to geometrical shapes. They could identify a given shape as a square, triangle, rectangle or a circle but their understanding of shapes did not include relationships among them or their properties.

After exposure to the applet, their internal representations of the geometrical shapes were extended to include their properties as well. On the other hand, their external representations, which were earlier restricted to drawings on paper were now more focused on properties and relationships. Prior to using the applet, when asked "what is a rectangle?" the grade 2 students would be able to draw a sketch of a rectangle on paper or say that "a door is a rectangle". But now their external representation of a rectangle was extended by their observation "the two triangles can be joined on the longest sides to form a standing rectangle". It is our belief that such DGE based mathematical tasks can help students focus on properties and relationships of geometrical shapes and thus enhance their external representations. The study shows that students, even as young as in grade 2, respond well while reasoning with geometrical objects via a digital applet when prompted appropriately and also communicate their thinking and reasoning. This is evident from the conversations (in the video) between the two children as they communicate their thinking related to the problem at hand.

The grade 6 students, however, were familiar with basic geometrical shapes and their properties. Prior to using the applet, their external representations were based on paper pencil drawings and recognition of properties of rectangles, including the rules for their area and perimeter. During the geometrical reasoning episode described in this article, their external representations toggled between the GeoGebra applet and their paper-pencil algebraic manipulations. Moving back and forth between the two representations fostered their geometrical as well as algebraic reasoning. They used the geometrical representation of the applet to justify and inform their paper-pencil algebraic calculation. After arriving at the equation $y=(3/2)x$, they justified it in terms of the dimensions of the darker rectangles. For both second and sixth graders, the process of making conjectures followed by providing reasoning and explanations appeared to come naturally. We consider this as an important aspect of the study and it reinforces the results by King and Schattschneider (1997), which posits that while a DGE cannot actually produce proofs, it can motivate the desire for proof in students.

In the case of the 6th graders, the applet helped them to conjecture that the positions of the midpoint lie on a straight line (the diagonal of the larger rectangle). However, when it came to justifying this conjecture, the students resorted to algebra and arrived at the equation of the straight-line $y=(3/2)x$. The students demonstrated a 'leap' in their thinking when they arrived at this equation after performing the required algebraic manipulation. It was also an 'aha' moment for the researchers to see students arrive at this unexpected result and also attempt to justify the same with practically no help or scaffolding from a teacher.

Secondly, we have deliberated on the role of GeoGebra as an amplifier and reorganizer in supporting students' geometrical explorations (research question 2). As an amplifier, the dragging feature of GeoGebra enabled students to quickly try out different solutions. The grade 2 students moved back and forth between the applet and their verbal explanations while the grade 6 students toggled between the applet and providing justifications in paper-pencil mode. The dragging feature was particularly useful as an amplifier.

The grade 2 students could drag the shapes and place them in the big square trying out multiple solutions. The grade 6 students could drag the blue midpoint and vary the sizes and measurements of the rectangles. GeoGebra also acted as a reorganizer of students' mental activity, as it helped to support their intuitions and enabled them to provide arguments and explanations. As a reorganizer, GeoGebra also offered many possibilities throughout the tasks, giving students access to higher-level concepts. For example, the grade 2 students explored geometrical objects and their properties even though they did not have an adequate classroom exposure to geometrical shapes. Prior to attempting the task, they only knew the names of geometrical shapes such as rectangle, square, triangle etc. However, after attempting the task they were able to argue and reason about the shapes and their interrelationships.

The discovery of a "standing rectangle" by joining two right triangles on the hypotenuse is one example. The applet presented the problem in the form of a game, in which, as they played along, they justified each step to each other and to their teacher. The grade 6 students arrived at a conjecture regarding the position of the midpoint inside the rectangle, leading to equal areas of the smaller rectangles. Having made the conjecture regarding the possibility of infinitely many positions on a *line* they went about the process of justifying it using algebra. They also obtained the equation of the line as $y = (3/2)x$ and made connections between this equation and the dimensions of the rectangles presented in the applet. Thus, the applet gave them access to drawing connections between geometric and algebraic representations, which is usually not expected at the level of grade 6. Hence, both episodes provide ample evidence of GeoGebra playing the role of an amplifier as well as a reorganizer of mental activity. In particular, the dragging feature of the GeoGebra applets worked as an amplifier and facilitated the process of making conjectures. However, in general, GeoGebra also allows the user to simultaneously view mathematical objects geometrically (in the graphics screen), symbolically (in the algebra view) and numerically (in the spreadsheet mode). This ability to use multiple representations can indeed enable the process of making conjectures.

Finally, the GeoGebra based applets appear to have characteristics of the three aspects of fidelity, namely mathematical, cognitive, and pedagogical (research question 3). Feedback taken from the participating children after the tasks were completed revealed that they found GeoGebra applets *easy to use*, *accurate* and *less time consuming*. One of the sixth graders commented that *dragging* made the problem *easy to think about*. As far as these students are concerned, the applets were easy to use, encouraged their participation and helped to augment their learning and thus have pedagogical fidelity. The applets also accurately represent the mathematical aspects of the problems they are designed for and hence also have mathematical fidelity. The cognitive fidelity aspect is also satisfied as the applets encourage looking for patterns and making and testing conjectures. These aspects of fidelity are important considerations while designing applets for student exploration.

Thus the findings suggest that such applets enhance students' internal and external representations by making them more dynamic and flexible. If designed on a dynamic software such as GeoGebra, such applets act as amplifiers and reorganizers of mental activity, giving students access to different ways and levels of thinking. Also the study shows that fidelity aspects play a critical role while designing applets to foster students' mathematical thinking.

Pitfalls or Limitations of the Study

Despite the encouraging findings, this study has a few limitations. One of the limitations is perhaps the fact that only two pairs of students have been observed through the videos. It is possible that if more pairs of students were selected the researchers might have been privy to a greater variety of responses and different levels of mathematical thinking on the part of students enabled by the GeoGebra applets. However, the research objectives of the study were primarily focused on the role of GeoGebra based applets in eliciting students' mathematical thinking and problem solving skills. At the time of selecting the video clips, the two described in this article seemed most appropriate to address the research questions of the study. Another limitation of the study is the lack of an in-depth follow up interview of the students after the tasks were completed. Time constraints permitted the researchers only a brief discussion with the students in which they shared their experience and feedback on working with the applets. An in-depth interview would have elicited more aspects of students' thinking in a DGE environment.

CONCLUSION

However, despite the limitations, our findings suggest that students in compulsory school with no prior experience on a DGE, can engage with geometrical problem solving in a meaningful way. We see evidence of these in both the grade 2 as well as grade 6 students. This leads us to conclude that DGE based applets, if designed and used appropriately, can contribute to developing mathematical thinking in young children and this is perhaps the most significant contribution of our study. It also makes an important point regarding the use of DGE in mathematics education. DGEs, such as GeoGebra, have many sophisticated features, which may be suitably exploited to enable students to explore mathematical concepts. However, such explorations may need a significant knowledge of the features and tools of the software and may not be suitable for young children. Instead, easy-to-use applets, such as the ones described in this article, do not require sophisticated knowledge of the software and may be attempted simply by using the dragging tool. These applets included features, which relate to the three aspects of mathematical, cognitive and pedagogical fidelity and are important considerations while designing digital applets for student exploration. This study also illustrates, that investigatory tasks, which require animation for discerning a pattern (thereby leading to conjecture making) are amenable for exploration via DGE. In this regard the grade 6 task, is more suitable for exploration as it allows the student to drag and vary the midpoint, thereby varying the areas of the darker rectangles, while the required solution points lie on a straight line. This has implications with regard to teacher preparation as well. Teachers need to be trained to create and develop mathematically rich tasks, which may be converted into DGE based applets, and thus become easily accessible to a large number of students.

Another important contribution of our study is that it has illustrated the role of GeoGebra based applets in eliciting students' mathematical thinking both at the primary school *and* middle school level. This is perhaps unique to our study. Most studies related to the Dynamic software focus on the impact of the tool usage on students of a specific grade level. This study however cuts across grades in elementary school and illustrates the fact that appropriately designed applets in a DGE have the potential to elicit and augment students' mathematical thinking and reasoning. It would be interesting to explore the potential of such DGE based applets in enhancing students' mathematical thinking in higher grades, especially in secondary school. The opportunity rendered by such applets in developing understanding in topics such as calculus and trigonometry needs to be explored and can provide scope for further research in this area.

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