Journal of Mathematics and Science Teacher
2024, 4(1), em054
e-ISSN: 2752-6054
https://www.mathsciteacher.com/

MODESTUM

Research Article **OPEN ACCESS**

Subject competency and teacher knowledge: An exploration of second-year pre-service mathematics teachers' difficulties in solving logarithmic problems using basic rules for logarithm

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Citation: Okoye-Ogbalu, I., & Nnadozie, V. (2024). Subject competency and teacher knowledge: An exploration of second-year pre-service mathematics teachers' difficulties in solving logarithmic problems using basic rules for logarithm. *Journal of Mathematics and Science Teacher, 4*(1), em054. https://doi.org/10.29333/mathsciteacher/13887

ARTICLE INFO

Received: 07 May 2023

Accepted: 30 Oct. 2023

ABSTRACT

Pre-service mathematics teachers' (PMTs) subject competency continues to engage scholars and researchers. Understanding level of knowledge of concepts that PMTs bring to their learning in university is crucial to developing their teacher knowledge. This article examines genetic decomposition of schemas PMTs in one university in South Africa build (to know about rules) for solving logarithmic equations. A mixed methods approach, and the action-process-object-schema (APOS) theory were employed to examine mental construction the 19 purposively selected PMTs that responded to a 90-minute simple logarithm research task (LRT) made while solving problems. Analysis of task scripts using percentage score forms the basis of the qualitative phase of the research. Individual interview was useful to elicit PMTs' views and perceptions of their encountered difficulties in solving LRT problems. One common difficulty was proving the logarithmic equation. This highlights gaps in PMTs' prior knowledge of logarithmic concepts and basic rules. Implications of the findings for PMT subject competency were discussed.

Keywords: APOS theory, teacher subject competency, logarithm, pre-service mathematics teacher, South Africa

INTRODUCTION

The South African Department of Higher Education and Training requires pre-service teachers (PSTs) to show subject competency in their subject majors to earn certification as a 'highly qualified teacher'. This is of particular importance in mathematics given the scenario, where high school mathematics teachers were not able to solve the level 4 mathematics questions posed in the national senior certificate (Matric) examination that required critical analysis and evaluation (Bansilal et al., 2014). However, the level of mathematical preparations in South African schools has been worrying. Recent reports continue to show decline in mathematics performance at school level (Dall, 2023; Fair, 2019; Taylor, 2021). Though the South African schooling system is uneven in terms of inequality and differences in educational outcomes (Spaull, 2019).

The current decreases in the number of students writing mathematics in Matric exams and in the exam pass rate that dropped from 58.0% in 2018 to 54.0% in 2019 is indeed a new twist to the trend (Shay, 2020). The combination of poor teaching and learning of mathematics (Pournara et al., 2015; Venkat & Spaull, 2015) and dwindling learner interest may explain the persistent and worsening trend of low Matric performances. But beyond that, it could be mirroring difficulties pre-service mathematics teachers (PMTs) face in learning mathematics concepts at university. It is important to continue to reassess ways to teach mathematics in university and evaluate if and what poses contradictions between school and university systems (Jooganah & Williams, 2016).

To develop PMTs' specialized content knowledge (Ndlovu et al., 2017) and thus improve preparation in terms of mathematical knowledge for teaching (Hine & Thai, 2019) of mathematics for secondary or high school mathematics, particularly in South Africa (Alex, 2019; Peng & Smida, 2015), attention must be given to supporting them to cultivate crucial teacher knowledge and subject competency in university. Examining the level of conceptual understanding PMTs bring to their learning at university can be an important first step in this direction. What defines conceptual understanding could be contested (Meyer, 2018), but what is important is that it forms part of the five strands of mathematical proficiency at high school (Killpatrick et al., 2001). Moreover, it involves learning facts and methods with understanding and the ability to remember, use and reconstruct as well as organize what is learnt (Meyer, 2018). Furthermore, beyond the procedural knowledge, conceptual understanding is necessary for integration and functional understanding of mathematical ideas (Meyer, 2018; Rittle-Johnson, 2019). However, some PMTs could have ended up learning mathematical concepts at high school by following procedures without understanding the why and how of things. It is essential to know how PMT associates relationships between procedural knowledge and conceptual understanding as two important types of mathematics knowledge (Rittle-Johnson, 2019), as well as how they might use conceptual knowledge in determining solution strategies during problem solving (Rittle-Johnson, 2017).

Several useful studies using action-process-object-schema (APOS) theory focus on school mathematics education (Díaz-Berrios & Martínez-Planell, 2022; Nga et al., 2023). In South Africa in particular, studies examined the contexts of school mathematics education including Jojo (2019) and Taylor (2021). But there is no known study in South African context that uses APOS stages to understand PMTs' difficulties in applying logarithmic concepts. The purpose of the present study was to use APOS theory to analyze PMTs' conceptual understanding and the necessary mental construction they made while solving the logarithmic problems.

Logarithm is an aspect of algebra that have diverse applications (Ansah, 2016) including in trends in population growth, radioactive decay and compound interest (Ostler, 2013). Although information technologies such as calculators and computers and more recent applications of artificial intelligence can be versatile and useful computing devices, logarithm remain relevant as important tool in mathematics and the sciences (Qi et al., 2017). Its history and uses in mathematics date back to 1634 when the concept was developed by a Scottish mathematician by name John Napier (Smith, 2000). Even though the concept constantly changes (Villarreal-Calderon, 2012), logarithm uses include to compare, measure, forecast, explain, illustrate, and interpret values (Weber, 2016), and importantly, it can be useful to model quantitative relationships, whereby students are "supported in conceptualizing quantities, their relationships and how they vary together" (Kuper & Carlson, 2019, p. 2). Thus, as part of subject competency, logarithm form key aspect of useful PMT content knowledge.

This article begins by reviewing the relevant literature on PMT conceptual knowledge of logarithm and the use of prior knowledge in solving logarithmic problems. This is followed by discussion of the theoretical framework and research method employed in the study. Next, it discusses the findings and their implications, as well as the limitations of the study. It ends with the conclusion summarising the key findings.

LITERATURE REVIEW

In South Africa, logarithm is part of both core and technical mathematics. However, Okoye-Ogbalu (2019) suggests that in grade 12 (the final year of high school), logarithm is only taught as an inverse of an exponential function. But grade 12 learner should have learnt the concept of inverse function. At this level, there is an expectation that they learn problem solving and graph work involving logarithmic functions, the basic laws of logarithm and its applications to real life problems (Usiskin, 2015).

However, at the university level 1 mathematics, application of logarithm extends to new and more complex functions, including natural logarithm, composite functions, derivative functions as well as proofs of the logarithm laws. This creates gap between what learners learnt in high school and the expectations at university (Ansah, 2016). For example, the curriculum assessment policy statement (Department of Basic Education [DoBE], 2012) shows that the high school mathematics curriculum emphasizes exponents as prerequisite knowledge to solve logarithmic problems. But the concept of proof of logarithmic properties (Mulqueeny, 2012) is first introduced to students at university level.

Perceptions & Troubling Reputation of Logarithm

Logarithm has a reputation for being difficult (Dintarini, 2018). This is probably because of its presentation as the inverse of exponential functions (Weber, 2016). Students perceive logarithm as irrelevant and confuse exponential and logarithmic laws (Campo-Meneses et al., 2021) and they tend to skip steps or fail to understand the meaning of the concept. They are disconcerted when letters are placed in positions resulting to chaos in the definition when working with logarithm (Aziz et al., 2017). Hence, students often make mistakes because of misconceptions and errors in simplifying exponential expressions (Cangelosi et al., 2013). Likewise, students make mistakes in applying the laws of logarithm (Fermsjö, 2014). Mulqueeny (2012) ranks, among the six laws of logarithm, the first and second as second and third, respectively in terms of the frequency of mistakes students make applying these laws (**Table 1**).

Table 1. Six laws of logarithm

Law	Law name	Mathematical representation
Law 1	Multiplication law	$log_b xy = log_b x + log_b y$
Law 2	Division law	$log_b (x/y) = log_b x - log_b y$
Law 3	Power law	$log_b x^y = y log_b x$
Law 4	Change of base law	$log_b x = log\ x / log\ b$
Law 5	Log of base law	$log_b b = 1$
Law 6	Log 1	$log_b 1 = 0$

Note. b>1 & a>1 with *x* & *y* being positive real numbers

In addition to the outlined reasons, most college entry-level or level 1 mathematics courses at university use logarithm as symbols of manipulation (Kenney, 2005). This hardly promote logarithm application in other activities or structures despite their importance in calculus and beyond (Ailing & Bin, 2016).

Students' Experience of Difficulties in Learning Logarithm

Students experience many difficulties in learning logarithm, which include, for an example, the mistakes they make in manipulating logarithmic expressions (Espedal, 2015). Other difficulties might include experiences of misunderstanding the meaning of the logarithmic concept, issue of pervasiveness of inverse definition of logarithm, and expression of logarithm as numbers (Díaz-Berrios & Martínez-Planell, 2022; Dintarini, 2018; Frketic et al., 2018). At high school levels, there are also the challenges of students' poor prior exposures to logarithm (Okoye-Ogbalu, 2019) and teachers' poor ability to make explicit the meaning of the symbol (Espedal, 2015). However, teachers themselves may lack understanding of logarithm (Fermsjö, 2014).

Pre-Service Mathematics Teachers' Mental Construction of Algebra Concepts

Ndlovu and Brijlall (2017) investigated mental constructions made by PMTs when learning determinant concepts. The study aimed to contribute to APOS theory in terms of instructional strategies. The findings revealed that, except for the few working at an object stage, many of PMTs operated at the action and process stages. The findings further suggest that though PMTs were able to carry out procedures effectively, several were not able to construct the meaning of the concept (Ndlovu & Brijlall, 2017). An earlier study by Ndlovu and Brijlall (2015) examined the link between the mental constructions of PMTs when learning matrix algebra concepts and their preliminary genetic decomposition (Dubinsky & McDonald, 2001). The findings showed that, in most cases, the mental constructions agreed with the preliminary genetic decomposition (Ndlovu & Brijlall, 2015). However, beyond the development of algebraic concepts, a difficulty that many students in general encounter in learning logarithm is that it requires multiplicative thinking (Kuper & Carlson, 2020). To develop effective PMTs' subject content knowledge of logarithm, both procedural understanding and knowledge of the concepts are critical.

THEORETICAL FRAMEWORK

Learning theorists argue that learning should be an active, constructive activity that encourages students to explore and develop problem-solving abilities (Baker, 2021). With reference to mathematics, Tall (2013) highlights the need to understand students' level of comprehension of structures and how they progressively master them using perceptions, connecting meanings, and drawing relationships between operations, and developing a sense of reasoning. APOS learning theory (Dubinsky & McDonald, 2001) is a useful lens to understand how to espouse and learn abstract mathematics (Oktac, 2019). APOS theory is an extension of Piaget's work on reflective abstraction. According to Maharaj (2014), in APOS theory: *Action* implies that a transformation is first conceived as an action when it is a reaction to stimuli, which an individual perceives as external. This calls for specific instructions and each step of the transformation needs to be performed (Arnon et al., 2014). For example, PMTs at action conception of solving logarithmic equation could directly apply logarithm laws to simplify the equation. *Process* is a mental structure that performs the same operation as the action, but solely in the mind of the individual (Arnon et al., 2014). The individual can imagine performing the transformation without having to execute each step. For example, PMTs at the process conception of solving logarithmic equation can mentally simplify the equation and write down the answer. *Object* describes the process, where one becomes totally aware, realizes that alterations can act on that totality, and constructs such transformations in one's imagination. The individual then encapsulates the process into an object. A PMT at this stage can recognize the similarity between $\log 2$ and $\log \frac{50^{\log 2}}{2^{\log 5}}$ without an explicit instruction on how to simplify it. *Schema* refers to organizing and linking the many actions, processes, and objects of mathematical activity into a coherent framework. It enables the individual to decide whether the schema applies when presented with a given mathematical situation.

Ndlovu and Brijlall (2017) note that APOS theory assumes that an individual has to have the appropriate mental structures relating to action, process, object, and schema to understand a given mathematical concept. Consequently, they argue that mental structures need to be detected and followed by learning activities that develop them (Ndlovu & Brijlall, 2017). Thus, to enhance learning of a mathematical concept, PMTs' learning activities should be designed in such a way that they develop the construction of appropriate mental structures (Maharaj, 2014). Cuevas (1984) suggests that common issues to learning mathematics include learners' language ability. This assertion reflects in the context of South Africa, where most learners, especially in rural and under-resourced schools, might struggle with mathematical fluency (Coetzer, 2023; Taylor 2021) and learning mathematics in English as a second language (Robertson & Graven, 2020). At high school level, students might then experience difficulties of understanding, tendency to applying laws incompletely or selectively, and misrepresenting concepts. However, it is known that at pre-university level, students struggle and have misconceptions, for example, in their application of reasoning in formal and everyday tasks because of poor strategies to reason logically (Bronkhorst et al., 2019). To understand the difficulties PMTs encountered while solving logarithmic problems, their level of knowledge of logarithm concepts was explored using the tasks that evaluated their construction of appropriate mental structures. Using APOS as a lens (Weyer, 2010) enabled us to disentangle PMTs' misconceptions of logarithmic concepts in both their mental structures and the stages of development of their knowledge and competency in its application.

Preliminary Genetic Decomposition

To detail a genetic decomposition for logarithm concept, an action conception of logarithmic equation requires logarithmic equation calculation with a specific value to have a meaning. PMTs at process level can recognize logarithm as a function. At this level, they will be able to predict an output value without explicit instruction on input values. Seeing logarithm as an object means being capable of recognizing idea immediately and manipulate it as a whole without details. A schema of logarithm is attained by PMT once they know when it is appropriate to apply either logarithm or natural logarithm while solving logarithmic equations.

METHODOLOGY

The study was conducted in school of education at a research-led university in South Africa that offers undergraduate Bachelor of Education as well as postgraduate qualifications. Mathematics is one of the major specialization areas of Bachelor of Education programs in the school. An explanatory mixed-methods research strategy (Creswell & Plano Clark, 2017) was adopted to maximize the strengths of the quantitative and qualitative approaches (Creswell et al., 2011). The first phase of the research gathered quantitative data, and the qualitative research data was collected using individual interview in the second phase (Bowen et al., 2017). Creswell and Plano Clark (2017) note that in explanatory mixed-method research, the researcher can explain or build on initial quantitative results using qualitative data collected at a later stage.

19 PMTs, purposively selected from the 150 first year PMTs enrolled in the mathematics method 2 class that were invited to participate in the study. These were among those that met the criteria for the selection that includes:

(a) They were studying mathematics as a major.

(b) They were in the further education and training phase.

(c) They attended lectures for a university level 1 mathematics class that must have covered topics in basic mathematics including on logarithm.

Together, the criteria enabled selection of participants that were studying towards becoming senior high school mathematics teachers and were the first 19 to indicate their willingness to participate.

In the first stage of quantitative data collection, simple logarithm research task (LRT) activity was given to the participants. The 90-minute activity comprised of four questions covering task items:

(1) simplification of logarithmic expressions through substitution of variables,

(2) solving a logarithmic equation,

(3) proving logarithmic equations, and

(4) solving a logarithmic equation by replacing some terms with a variable.

The use of simple LRT task design was informed by APOS theory and the primary purpose of the activity, which was to test if PMTs can solve logarithmic problems using knowledge of basic laws for logarithm from prior learning. Whilst seemingly overlapping, the tasks were purposefully designed to enable examination beyond procedural understanding, PMTs ability to use variable substitution of logarithm terms. Using their percentage score, LRT task scripts were analyzed for application of logarithm concepts with understanding. The reflections on the analysis guided the preparations in the second and qualitative stage of data collection.

The use of the individual interviews in the second phase was useful to solicit the mental construction of PMTs while solving LRT problems. Their detailed explanations clarified the information for data analysis (Bowen et al., 2017). Each interview lasted 45 minutes. The questions were based on extracts from the written responses in their LRT answers. Data convergence, achieved at methods level (Creswell & Plano Clark, 2017), further crystalize the analysis. Compliant with the ethical approval for conducting the interviews, pseudonyms are used to protect the participants' identities (Powney & Watts, 2018).

Each of the four LRT task questions forms the item for analysis. Question 1 explored PSTs' knowledge of simplifying logarithm expression; question 2 tested their knowledge on solving a logarithmic equation; question 3 examined their knowledge on how to prove equations stated in the logarithmic form; question 4 tested their problem-solving skills, understanding of the relationship between concepts, and application of their knowledge and procedures in solving the problem; and further required them to sketch a logarithmic function, which involves their applying conceptual knowledge of logarithmic function. We compiled a tabular summary of the number of responses for each item and extracts of their written responses illustrate as examples. Next, analyzed written responses and the comments made by the participants during the individual interviews, categorized and ranked one to four per item 1 to item 4, are presented in the following.

FINDINGS

Item 1

Item 1 is defined, as follows: If $log_9 7 = A$ and $log_9 10 = B$, find the $log_9 810 + log_9 63$ in terms of A and B?

Table 2 shows the response categories for item 1.

Table 2. Response categories for item 1

Categories	1	2	3	4
Indicator	No answer written or incorrect solution	Expanded 810 & 63 as a product of nine & a number	Applied logarithmic law for multiplication	Simplified & wrote down correct answer
Number of responses	15	4	1	1

The results show that, in category 1, 15 of the 19 participants did not write any answer or wrote an incorrect solution in the item 1 question. In category 2, only four participants were able to simplify 63 and 810 as a product of seven, nine, and 10. In category 4, just one participant was able to simplify and write down the correct answer to the logarithmic expression in item 1. These responses suggest that most of the participants did not know how to simplify the item 1 logarithmic expression.

Extract 1 (**Figure 1**) and extract 2 (**Figure 2**) show examples of the responses. Also, three of four respondents that responded in category 2 showed that they had no idea of the laws of the logarithm, as Aphi's written response for item 1 shows (**Figure 1**).

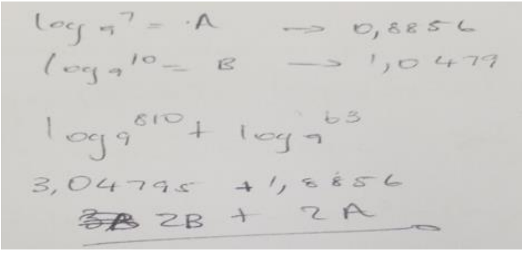

Figure 1. Extract 1: Aphi's written response for item 1 (Source: Student's LRT answer sheet)

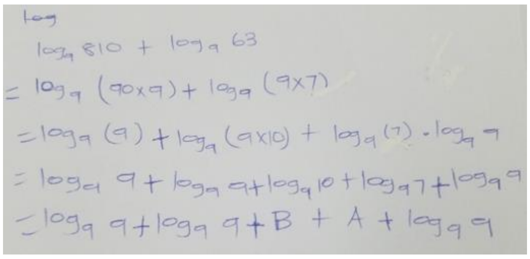

Figure 2. Extract 2: Amu's written response for item 1 (Source: Student's LRT answer sheet)

Common errors in response to the item 1 logarithmic expression ranged from writing 63 as a product of seven and nine to misapplying the logarithm law in simplifying the expression. In expanding 810 as a product of the given variables, one respondent in category 2 used 90 and nine, and then later expanded 90 as a product of 10 and nine. In contrast, other respondents in this category expanded 810 as a product of 10 and 81 but failed to apply the correct law of the logarithm. Aphi's response is an example of incorrect simplification of the item 1 logarithmic expression. As extract 1 shows, Aphi did not correctly expand 810 as the product of nine and 10, or 63 as a product of nine and seven. Rather, she seems to have made use of a calculator to obtain the value of $log_9 810 + log_9 63$. To illustrate this assertion, in line 4 of Aphi's written response, one can tell the value of the expression, but the item 1 task activity requires the respondent to find a solution, that is, simplify in terms of *A* and *B*. In the fifth line of Aphi's response, it is possible to infer that she understands that the answer to the question must be in terms of *A* and *B*. However, the written response shows that Aphi could not simplify the expression to arrive at the correct answer.

In the interview with Aphi, we sought to clarify her difficulties with simplifying.

> Researcher: It seems you used a calculator to evaluate $log_9 810$ and $log_9 63$; so, do you think that $2B$ is equal to 3,04785, as you have written in your task?
>
> Aphi: I was just trying to get my answer in terms of A and B as the question indicated. I used my calculator to press the value of that, then put A and B.
>
> Researcher: How do you decide which term would have *A* or *B*?
>
> Aphi: I just guessed; I did not know at all. I was trying to get my answer in terms of A and B.

The respondents' responses in category 2 indicate that they have some idea of how to simplify the logarithmic expression because they provided correct and complete set simplification in expanding 810 as a product of nine and 10, and 63 as a product of nine and seven. Amu's written response for item 1 illustrates, these respondents applied the right steps, but failed to apply the multiplicative law of the logarithm correctly.

Amu's response in category 2, shows that she applied an incorrect logarithmic law in the simplification of the logarithmic expression. In the expansion of 810, she started by expressing 810 as 90×9 and then further expanded 90 as 9×10. This shows that she was still at the action conception stage in the simplification of the logarithmic expression. She needed to continue with the correct application of the logarithmic law in the next part, which is step three. Her inability to do so shows a lack of understanding of the logarithmic laws. Again, in the fourth step, she equated $log_9 7 \times log_9 9$ with $log_9 7 + log_9 9$. This could be because she mistakenly misapplied the law. However, it may also be that she did not have the conceptual understanding of the applicable multiplicative law of logarithm.

In the interview, Amu explained why she adopted this approach.

> Researcher: In the third step of the solution, you wrote $log_9(7 \times 9)$ as $log_9 7 \times log_9 9$ and then in the next step as $log_9 7 + log_9 9$. Can you explain how you got to the fourth step?
>
> Amu: I think I was just trying to simplify the big numbers in terms of nine, seven, and 10. To be honest, I was confused about how to use the law of log correctly.

Researcher: Could that be why you did not continue?

Amu: That was why I could not continue. I was not sure if I was simplifying the sum correctly.

Researcher: Do you know that the log of any number to its base is one?

Amu: I guessed it, but I cannot say for sure.

Amu was unable to show she understands that solving the question meant simplifying it fully without leaving any term in logarithm form. That means substituting $log97$ with A and $log910$ with B. By not knowing that $log_9 9 = 1$, she inconsistently applied the logarithmic law. This illustrates her lack of understanding of its application, which affirms Naidoo and Naidoo's (2007) observation that PMTs lack conceptual understanding of mathematical concepts. However, Amu was able to correctly substitute the variables for both $log_9 7 = A$ and $log_9 10 = B$, which suggests that though she understood the question, she was unable to follow the rules properly to arrive at the answer.

Item 2

Item 2 is, as follows: Solve for x: $log_2 x + log_2 5 = 3$.

The summary of scores for the responses to item 2 is presented in **Table 3**.

Table 3. Response categories for item 2

Categories	1	2	3	4
Indicator	No answer written or incorrect solution	Applied logarithm law of multiplication	Converted log into exponential form	Solved & wrote down correct answer
Number of responses	15	4	4	3

Item 2 assessed whether the respondents could solve a logarithmic equation that involves a simple linear equation using the rules for logarithm. Category 1 in **Table 2** shows that some could not apply the logarithm law of multiplication. This means that these respondents had not reached the action conception stage of solving logarithmic equations. Zee's written response for item 2 shows that one respondent in category 1, Zee, applied the change of base formula using a natural logarithm (**Figure 3**).

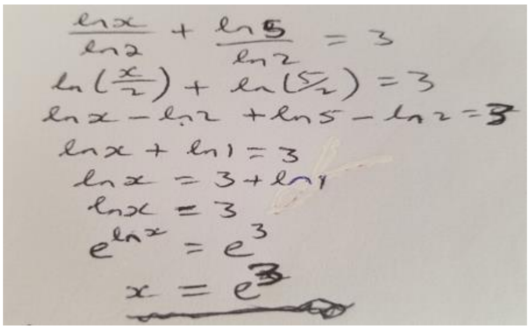

Figure 3. Extract 3: Zee's written response for item 2 (Source: Student's LRT answer sheet)

Zee seems confused about the difference between a logarithm and a natural logarithm (a natural logarithm is different from a logarithm because it is a logarithm to base *e*). While she appears to have tried to apply the change of the base law, she did so with a natural logarithm. Her response suggests that she had difficulty reaching the action conception stage in solving the logarithmic equation. In her second step, Zee does not seem to know the logarithm law of division since she assumes that $\frac{\ln x}{\ln 2}$ is the same as $\ln \frac{x}{2}$. In step three, while she has an idea about the division law, it appears that she did not know when and how to apply it. Thus, she did not solve the equation correctly because of poor knowledge or application of the concept of the logarithm.

In the interview with Zee, she clarified the difficulties that led to these errors.

Researcher: Do you think that the second step of your solution is equal to the change of base that you did in your first step?

Zee: I am not sure. I did that so that I could apply the subtraction law of log.

Researcher: Do you know the difference between a logarithm and a natural logarithm?

Zee: I do not remember, but I prefer to use "*ln*" to "*log*" when I am doing change of base so that in the end, I can quickly introduce "*e*" to get rid of the "*ln*".

Zee's response suggests that she lacks conceptual understanding of logarithmic concepts. Consistent with this finding, Matz (as cited in Siyepu, 2013), suggests that this type of errors that, for an example in this case Zee made, persist because of rote learning that prevents an individual from engaging with logarithmic concepts and its meaning. This lack of conceptual understanding that might result from rote learning typifies a surface approach to learning (Jojo, 2011; Nga et al., 2023).

In category 3, all the respondents except Mpho correctly applied the logarithm law of multiplication. As Mpho's written response for item 2 illustrates, Mpho used a different method and converted $\log_2 5$ to a decimal number, then subtracted it from both sides of the equation (**Figure 4**).

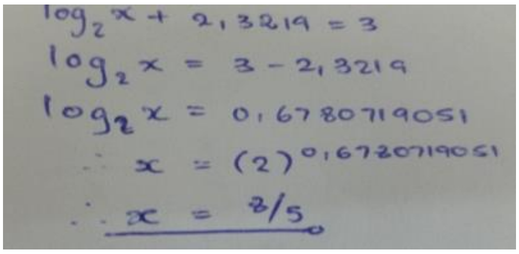

Figure 4. Extract 4: Mpho's written response for item 2 (Source: Student's LRT answer sheet)

This response shows that Mpho was at the action conception stage of solving logarithmic equations. She would have avoided simplifying the left-hand side of the equation because she used a calculator to evaluate $\log_2 5$. Yet, the question was meant to test the respondent's ability to apply the multiplicative rule for logarithm in solving the equation without resort to use of calculators. Thus, Mpho's response implies she was unable to advance to the process conception stage.

Item 3

Item 3 is defined, as follows: Prove that $\log\left(\frac{50^{log\,2}}{2^{log\,5}}\right) = log\ 2$.

Table 4 shows the response categories for item 3. The responses in category 1 show that the respondents had difficulties in proving some logarithmic equations. In category 1, seven of seventeen respondents provided no answer, and the rest solved the question incorrectly. This points to a lack of prior knowledge of the laws of logarithm. Patu's written response for item 3 illustrates that, some respondents like Patu divided the base 50 by two (**Figure 5**).

Table 4. Response categories for item 3

Categories	1	2	3
Indicator	No answer written or incorrect solution	Applied logarithm quotient law	Solved to get right-hand side of equation
Number of responses	17	2	1

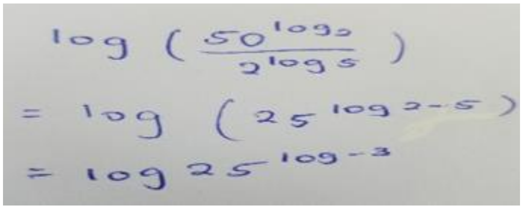

Figure 5. Extract 5: Patu's written response for item 3 (Source: Student's LRT answer sheet)

Patu treated 50 and two as the coefficient of log 2 and log 5, respectively, which is why she got 25 in step 2. While she stated the action conception of the quotient law of exponent in line 2, she did not seem to know where or when to apply it.

The interview with Patu shed light in the difficulties she confronted.

> Researcher: How did you get 25 in the second step of your work?
>
> Patu: I just divided 50 by two, and it gives me 25 [silent] And I applied exponential law, that is why I have log 2 minus five.
>
> Researcher: Do you know the differences between the laws of logarithm and exponential laws?
>
> Patu: I cannot say for sure, but I know both the laws to an extent.
>
> Researcher: Why could not you go ahead with the solution after your third step?
>
> Patu: The sum was very difficult for me from there. I even entered it in the calculator, and I had errors.

Patu's responses point to her lack of conceptual and procedural knowledge of proving the logarithmic equation, which means that she had not reached the action conception stage necessary to advance to later stages.

In category 2, two of the respondents applied the logarithm quotient law, but one, Zimba was unable to continue correctly. As Zimba's written response for item 3 shows, she seems to have changed the logarithm to the natural logarithm (**Figure 6**).

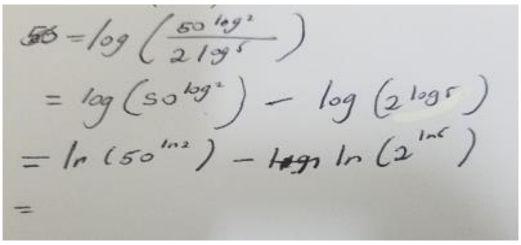

Figure 6. Extract 6: Zimba's written response for item 3 (Source: Student's LRT answer sheet)

In step 3 of her work, Zimba is not at the action stage, and she lacked the procedural knowledge required to continue to prove the expression.

In the interview with Zimba, she clarified, as follows:

> Researcher: Why did you change from logarithm to natural logarithm in your step 3?
>
> Zimba: I just wrote it down to see if I could proceed from there, but I could not think of anything to do from that step.
>
> Researcher: Do you know the differences between a logarithm and a natural logarithm?
>
> Zimba: I do not remember, but they are the same, I guess.
>
> Researcher: Do you know about the power law of logarithm?
>
> Zimba: Maybe, but I cannot say what it is right now.

Whilst Zimba's responses show that she knows the division law of logarithm, she was unable to advance to the action stage of the logarithmic equation because she lacked knowledge of the application of the power law of logarithm.

In category 3, only one respondent, Qwabe demonstrated conceptual knowledge of the laws of logarithm and the procedural knowledge of where and when to apply them. Qwabe's written response attests that she might have cognitively constructed the structure of proving logarithmic equations and was thus able to use the necessary laws to prove the equation. Her responses show that she was able to carry out the procedures, not only for the application of logarithm laws, but to factorize the expression so that it becomes easier to simplify further. As Qwabe's written response for item 3 illustrates, she was also able to apply the quotient law of logarithm in step 5 in the revert order, which shows that she is at the process stage (**Figure 7**).

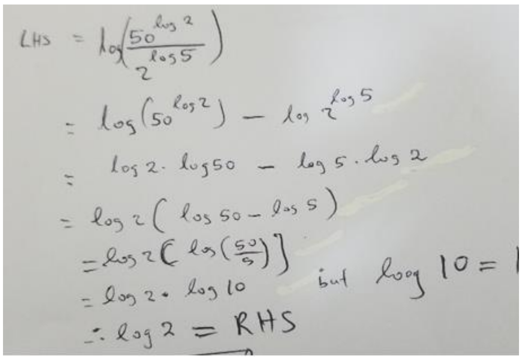

Figure 7. Extract 7: Qwabe's written response for item 3 (Source: Student's LRT answer sheet)

The fact that Qwabe was able to follow the steps correctly shows that she had both conceptual and procedural knowledge and application of the four laws of logarithm, which she applied correctly to prove the logarithmic equation.

Item 4

Item 4 is defined, as follows: Find the value(s) of x for which: $2\log_9 x + 6\log_x 9 = 7$?

Table 5 shows the response categories for item 4.

Table 5. Response categories for item 4

Categories	1	2	3	4
Indicator	No answer written or incorrect solution	Applied change of base law of logarithm	Solved quadratic equation	Checked for restrictions for values of *x*
Number of responses	16	3	2	1

The responses in category 1 show that most of the participants had difficulties in solving the question. Indeed, four of the 16 respondents seemed unable to proceed as they did not write anything in the space provided for the response. The remaining 12 attempted the question, but incorrectly. This suggests that they were unfamiliar with the change of base law of logarithm. Again, as Zika's written response for item 4 illustrates, those who tried to solve the question were simplifying it in the wrong way (**Figure 8**).

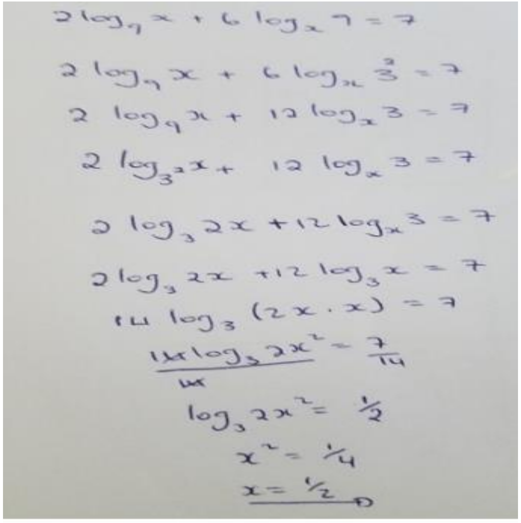

Figure 8. Extract 8: Zika's written response for item 4 (Source: Student's LRT answer sheet)

In Zika's response to item 4, she seems to attempt to keep the equation in base three. In step 3, she applied the power law of logarithm correctly, which could suggest she probably was aware of the laws of logarithm but failed to apply them correctly because she did not know when and where each law applies. In the interview Zika clarified, as follows:

> Researcher: Why did you change from nine to 3^2 in step four?
>
> Zika: I thought that changing the base to three would help me apply the change of base formula for logarithm. I can easily change them all to base three.
>
> Researcher: Is that the reason why you change base x to base three in the second term of your step 6?
>
> Zika: Yes. You see, with that, I can then apply the other law so that I can simplify the sum.

This response infers that Zika struggled with the problem because she had not fully developed the action conception stage of the logarithmic concept.

In category 2, three respondents applied the logarithm change of base law, but two of these could not proceed correctly. As Maza's written response to item 4 shows, they changed the logarithm to the natural logarithm (**Figure 9**).

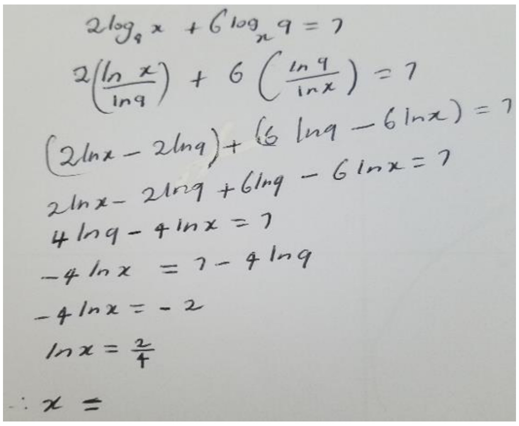

Figure 9. Extract 9: Maza's written response for item 4 (Source: Student's LRT answer sheet)

In step 3 of Maza's response, she seems to be applying the logarithm quotient law. This being the case, she was not simply making a mistake, but was struggling to construct meaningful knowledge of the learned concept and was thus unable to differentiate between the logarithm and the natural logarithm. This suggests that she was unable to continue to solve the equation because she did not have the procedural knowledge to do so. In the interview with Maza, she clarified, as follows:

> Researcher: I can see that you changed log to natural log again here. Do you think step 2 is equal to step 3 in your solution?
>
> Maza: I guess it is … is it not part of the log laws?

Researcher: No, it is not the correct application. Assuming your steps are right, why could not you write down the value of *x* at your final step?

Maza: I forgot how to manipulate "*ln*."

Maza's response shows that she was aware of the law of the logarithm but was not sure how to apply it. She had not fully developed the action conception stage of the logarithmic equation.

Xolo was the only respondent that solved the equation from category 1 to category 4. As Xolo's written response for item 4 illustrates, she shows conceptual knowledge of the change of base law of logarithm as well as procedural knowledge of how and when to apply it. Xolo's response suggests prior knowledge of the change of base law of logarithm. She shows understanding of the relationship between the logarithmic concepts and the quadratic equation as she formed and solved the quadratic equation without using the variable substitution of any term like substituting $\log x$ to be K, as an example. Although she appears unable to check for the restrictions for the correct values of x, Xolo encapsulated the action into a process, which meant that she was able to progress from action stage to object stage (**Figure 10**).

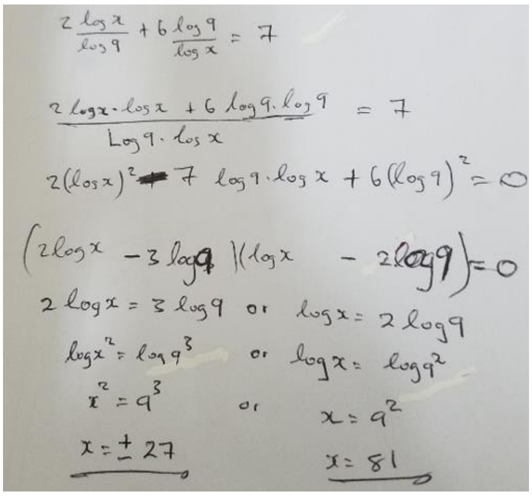

Figure 10. Extract 10: Xolo's written response for item 4 (Source: Student's LRT answer sheet)

Again, as Xolo's written response for item 4 shows, she correctly performed the necessary action in the solution of the equation process. This implies clear understanding of the change of base law of logarithm and its application to solve a logarithmic problem. However, all the respondents, including Xolo, did not apply the variable substitution of logarithm term in solving the equation. Such omission shows their lack of knowledge of the use of the variable substitution of logarithm term method.

DISCUSSION

The findings show that PMTs' knowledge of logarithm is at the action stage of APOS. This means that their responses to LRT task items show weak understanding of the calculation rules for logarithm. The patterns of the miscalculations seen in their task responses point to the difficulties that majority of PMT participants in the study encountered. These difficulties in understanding logarithm and their concepts show as well in their other APOS stages of solving LRT task. However, their difficulties in working with calculation rules for logarithm is reflective of weak schooling backgrounds in mathematics discussed above. In turn, the lack of acquaintance to the basic rules for logarithm couple with the discontinuity between what they learnt in high school and at university level 1 mathematics to result in further difficulties in solving logarithmic problems using these rules. Together, these difficulties are replete of possible mathematics anxieties (Bekdemir, 2010) that PMTs might have that highlight the extent of their challenges. Even though at second year of their study in university, they still faced the difficulties that impeded their understanding and applying logarithmic concepts at previous levels. Hence their challenges were more noticeable in tasks that require solving logarithmic problems and that involve application of the laws of logarithm.

PMTs were not confident in applying the various laws of logarithm as their written responses to LRT tasks clearly illustrated. While some of PMTs state the applicable law of logarithm, as exemplified in extracts 2, 3, and 5, they did not proceed to solve the problem. This reveals lack of understanding of the conceptual evolution of the logarithmic concept. Again, while some of PMTs' responses show some knowledge and understanding of the laws, they were unable to carry out the procedures. For example, in extract 1 and extract 8, the respondents could not apply the laws of logarithm correctly in item 1 and item 4, which involve action construction. This shows deficiencies in procedural knowledge of solving logarithmic problems using basic rules for logarithm. They also demonstrated interiorization of action in some of their responses. For example, Qwabe used the correct logarithm law in solving the problems in item 3 and made the connection between the different laws of logarithm. In similar instances, some respondents accurately drew the logarithmic graph, but did so as the inverse of the exponential graph along $y = x$. This implies incomplete understanding of the procedure involved in plotting the logarithmic graph.

Wasserman (2016) argues that learning a concept involves building cognitive structures around the name to support its meaning and use. Tall (2013) notes the importance and relevance of developing students' understanding of mathematics using levels of understanding of structures and how they progressively master such. He further emphasizes the use of students' perceptions, their ability to connect meanings and draw relationships between operations. This also involves improving mathematical thinking (Celik & Ozdemir, 2020) and developing a sense of mathematical agency (Schoenfeld, 2020) and reasoning to improve their understanding of mathematics (Stewart et al., 2019).These assertions are in consonance with Ndlovu and Brijlall's (2017) contention that APOS theory assumes that a person requires appropriate mental structures that relate to action, process, object, and schema to understand a mathematical concept, which is consistent with the findings of the present study.

Furthermore, PMTs' difficulty in understanding the logarithmic concept resulted in inability to differentiate the laws and outright misapplication of logarithmic concepts in their steps to solve the logarithmic problems. Viewed from the perspective of APOS stages, the difficulties in solving LRT tasks show a lack of mental structures that relate to understanding. PMTs' many misapplications, for examples in Patu's and Zimba's written responses in extract 5 and extract 6 above, demonstrate this. Similar conclusions by Dubinsky and McDonald (2001) point to difficulties in learning of algebra that students encounter because of poor understanding and poor prior knowledge of other mathematical concepts. Yet, there seems to be an assumption that PMTs enter university with apposite prior knowledge of logarithm. The findings of the present study suggest that such assumption is simplistic. Granted that their arithmetic and high school learning of algebra should enable PMTs to generalize and formulate new knowledge, however, in the present study context, the findings prove that it is not always the case. Thus, it is difficult to see how such learning could have prepared PMTs for the procedural and conceptual understanding needed to deal with learning logarithm at university. About 94.8% of all the responses to LRT tasks revealed difficulties in solving the logarithmic questions. In almost all the task items (1 to 5), the respondents were unable to correctly apply the logarithm concept and progress to complete answers. As item 4 illustrates, many experienced the greatest difficulty in the change of the base question and the use of the variable substitution of logarithm term.

While the literature provides a broad overview of the difficulties students confront in solving logarithmic problems and relates them to mistakes in manipulating logarithmic expressions and to challenges in understanding the meaning of the logarithmic concept (Dintarini, 2018), the current study's findings show that incorrect notions of logarithm, due to poor prior understanding and knowledge, also contribute to such errors. Aziz et al. (2017) aver that students make common mistakes due to misconceptions about logarithm, arithmetical problems, and misuse of algebra concepts. This is instructive to explain why, beyond respondents that wrote wrong solutions, some of PMTs still were unable to write any responses to the task items on their LRT answer sheets.

Collectively, the findings imply that bridging PMTs' prior knowledge gap and equipping them to understand and apply logarithm are critical challenges for the mathematics teacher education in South Africa. Logarithm is an aspect of PMT content knowledge that should receive close attention especially because of the impression that learners have of their esoteric nature. Usiskin (2015) asserts that for learners to understand mathematics, they need to be aware of its use in real life and how it applies to other concepts or areas. Hence beyond procedural knowledge, PMT's conceptual understanding should be stressed. For an example, it is not enough for PMTs' to know how to solve logarithmic equations. As high school mathematics teachers in the making, it is important to prepare them to cultivate the necessary subject competency and develop teacher knowledge. This must include advancing their knowledge of the application of logarithm to solving problems using basic rules, as well as their ability to apply them to concepts such as finance, sequence, calculus, and other areas.

Limitations of the Study

Whilst the study brings a refreshing contextual insight on the issues around the intersections of prior knowledge from high school and level 1 university knowledge of mathematical concepts, a cross-sectional study that includes other levels of study than second year PMTs might produce more nuanced results. However, the study was designed to be a small-scale investigation of the phenomenon to elicit further research interest. Though we recognize that a different type of test would have clarified some of the questions that the students' lack of responses leaves open, and we acknowledge a drawback in the use of inferential analysis, LRT tasks' design limits an in-depth analysis. It would seem hard to say whether those students with correct responses simply were better at learning the procedures. However, the choice of simple LRT procedural task was deliberate as opposed to tenuous. The aim was to elicit important information useful to examine and analyze PMTs' ability to engage, beyond the procedure, their use of conceptual knowledge and understanding of application of calculation rules in making connections to prior learning.

CONCLUSIONS

Logarithm is an important mathematical concept, and the mathematics teachers' subject content knowledge of logarithm is thus crucial. Continuing to improve PMTs' competency in and applied knowledge of logarithm calls for a focused effort to enhance their mental construction of concepts. An important step in doing this is to first understand the difficulties that impede such conceptual knowledge.

The findings of the study point to the importance of finding ways of teaching logarithm to PMTs that must take into consideration their prior knowledge gap. In high school mathematics, teachers treat logarithm superficially as an inverse of exponents, while at university, it involves complex and deeper knowledge. In addition to generic level 1 mathematics in university, there exists a need to target PMTs with bridging programmes at entry level in form of instructional support combined with specialist tutoring. This is essential to augment their knowledge and application of mathematical concepts, and in this specific case, logarithm concepts. In the context of South Africa, this augmentation is necessary in order to undo the cyclic trend of poor performances in high school mathematics. Partly associated to poor teaching and learning of the subject, deficient teacher subject competency in high school teaching of mathematics manifests in the type of example in the study that PMTs show at university. If unchecked, the trend could continue to replicate misconceptions of application of logarithm and indeed the worrisome level of mathematical preparations in schools.

Author contributions: All authors have sufficiently contributed to the study and agreed with the results and conclusions.

Funding: No funding source is reported for this study.

Ethical statement: The authors stated that this study was approved by the Humanities & Social Sciences Research Ethics Committee. Protocol reference number: HSS/0347/018M. All participants gave their informed consent to be part of the study before they participated in the research.

Declaration of interest: No conflict of interest is declared by authors.

Data sharing statement: Data supporting the findings and conclusions are available upon request from the corresponding author.

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