Journal of Mathematics and Science Teacher
2026, 6(2), em102
e-ISSN: 2752-6054
https://www.mathsciteacher.com/

MODESTUM

Research Article **OPEN ACCESS**

Tracking mathematics teachers' professional vision in a practice-based professional leaning setting

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Citation: Naik, S. S. (2026). Tracking mathematics teachers' professional vision in a practice-based professional leaning setting. *Journal of Mathematics and Science Teacher, 6*(2), em102. https://doi.org/10.29333/mathsciteacher/17943

ARTICLE INFO

Received: 06 Dec 2024
Accepted: 16 Dec 2025

ABSTRACT

This study investigates shifts in mathematics teachers' professional vision within a practice-based professional learning setting where teachers co-planned, observed, and reflected on live teaching in a laboratory classroom. Findings illustrate how teachers' noticing evolved from evaluative comments to interpretive attention toward students' mathematical thinking, recognizing and leveraging "kids' mathematics" as a pedagogical resource. Teachers' participation transitioned from individualistic "I/you" framing to a collaborative "we," demonstrating deeper collective engagement with instructional planning and content-specific pedagogical reasoning. Their discussions shifted from fragmented, surface-level observations to rich, sustained analyses leading to concrete, collaboratively developed instructional tools for subsequent lessons. These shifts were evident in increased attention to students' ideas, improved knowledge-based reasoning, and production of mathematically dense teaching resources. The study highlights how practice-based, collective examination of actual teaching enables teachers to develop an analytic mindset, supporting systematic unpacking of students' thinking and improving content-specific instructional decisions essential for mathematics classrooms.

Keywords: professional vision, laboratory setting, practice-based professional development, noticing student ideas, pedagogical content knowledge

INTRODUCTION

The recent national-level study by Haug and Mork (2021) on effective teacher education highlighted two key insights for 21st century teacher education:

(1) Teachers highly value the modeling of instructional strategies, and

(2) Collective participation in professional development, also emphasized by Desimone (2009), positively influences student outcomes.

The early focus on practice-based professional development (Ball & Cohen, 1999) led to the widespread use of artefacts of practice as tools for teacher learning, providing a space for modeling instructional strategies and reflective engagement. Many research studies that claim significant teachers' learning from reflections based on representations of practice nominate videos of teaching as an authentic record of practice (Borko, 2004; Santagata, 2011). In their seminal cross-country review across 14 nations, Popova et al. (2022) defined effective teacher education as that which significantly improves student performance, identifying exposure to "subject-specific pedagogy" as a key indicator of impactful professional development. These suggestions prescribe that analyzing and reflecting on teaching (content-specific) is a valuable activity that has the potential to foster teacher learning (Sherin & Han, 2004) and also the one that opens up a window into classroom dynamics that conveys "the complexity and subtlety of classroom teaching as it occurs in real time" (Brophy, 2004, p. 287).

Watching live teaching is much less common in teacher professional development, which is ironic, as the school itself could channel access to live teaching. Although the research is somewhat limited, observing live lessons has been shown to have the potential to enhance teachers' professional learning, mainly by bringing changes in their beliefs and practices (Grierson & Gallagher, 2009; Saphier & West, 2010). Most of these studies use "demonstration classes" to refer to "live teaching settings" for teacher professional development. In these demonstration class studies, teachers reflected that "the modeling, observing, debriefing and problem solving were the most valuable components" of seeing the teaching (Butler & Cartier, 2004, p. 447). Teachers' engagement in actual teaching as a collective participation leads to situated learning, as the learning context is the same as the one in which it will be applied, making actual teaching observation an authentic activity for teacher learning (Greeno, 2003; Lave & Wenger, 2001; Putnam & Borko, 2000).

Teachers' learning around the artefacts of teaching, especially around videos of teaching is characterized differently among researchers. Often, teachers' beliefs, practices or knowledge gains are studied to characterize their learning. However, what teachers notice when they analyze teaching has been the root of the various characterizations of teacher learning (Van Es & Sherin, 2002). Gamoran Sherin and van Es (2009) introduced a measure called "professional vision", to describe the change in teachers' noticing. In a video club study, these authors illustrated how teachers' professional vision was improved when they were engaged with and reflected on videos of teaching over a period. They defined professional vision as an "ability to notice and interpret significant features of classroom interactions". In the said video club studies, they found teachers' professional vision changing when they invest time in talking about teaching. However, the characteristics of teachers' participation as a group and its connection with what is seen in the video still need more elaboration.

Professional learning discussed in the study is one of a kind. It is practice-based and facilitates – co-planning, watching and collaboratively reflecting on actual teaching. This triad of activities created a complex space for teachers' engagement with and around the teaching, leading to a shift in their professional vision. This shift is observed through changes in teachers' – participation, nature and ways of noticing students' ideas and their knowledge-based reasoning. In particular, the paper responds to the following questions.

RQ1 What topics and concerns around classroom teaching do the teachers notice?

RQ2 What is the characterization of teacher engagement in practice-based professional learning?

RQ3 What is the shift in teachers' professional vision - specifically in their knowledge-based reasoning about students' mathematical thinking?

The following sections summarize the prior research on teachers' professional vision and narrate the professional learning setting.

CONCEPTUAL CONSIDERATIONS

The recent review by Wei et al. (2023) positions noticing student mathematical thinking as central to learning to teach. However, research on teacher noticing has historically emerged from multiple distinct strands. The early work on teachers' noticing is informed by research on the nature of expertise. Building on the expertise ideas, Van Es and Sherin (2002) suggested the following for teacher learning:

1. Identifying what is important and noteworthy about a classroom situation;
2. Making connections between classroom interactions;
3. Using the context (school, students and content) to explain classroom interaction (Van Es & Sherin, 2002, p.573).

Van Es and Sherin (2002) slowly moved their focus on the importance of teachers' interpretation of classroom events that allow comprehension of students' mathematical thinking. Developing skills in interpreting classroom interactions and using these interpretations to inform their pedagogical decisions was the main focus of their studies.

The research paradigm on noticing emphasizes cognitive aspects, such as the individual's attention, and minimizes the social and situated nature of learning (Putnam & Borko, 2000; Van Es & Sherin, 2002). Notably, models of professional development that considered individual teachers' noticing in individualistic tasks undervalue the situated and collaborative learning aspects that would facilitate the noticing. Framing the learning process to notice as professional vision emphasizes the social nature of what teachers see and how they make sense of it. The conceptualization of the professional vision goes one step ahead of noticing as it focuses on changes in teachers' reasoning, whereas noticing studies focused only on the stance and topics of teachers' noticing. For example, the research done by – Ethel and McMeniman (2000), Zwart et al. (2008) and Talanquer et al. (2013) either studied the topic or stance that teachers notice.

Professional Vision of Teachers

What teachers notice in their teaching impacts their decisions, the learning opportunities they design, and subsequent lesson plans. The concept of "learning to notice" is not restricted to teaching and is present in most professions. Charles (1994), a linguist and anthropologist, explores how professional practices of seeing become socially recognized as different from and better than those of laypeople. He illustrates the concept of professional vision by investigating two contexts of professional activity: Archaeological field excavation and legal argumentation.

Sherin (2001) introduced "professional vision" as a measure to understand practising teachers' learning in educational research. Their work has influenced several to facilitate teachers' discussion on videos of teaching and track their noticing and reasoning of classroom interactions. As a result, the noticing construct has been studied widely, noticing while teaching, post-teaching, noticing in colleagues' work, and noticing through mediating tools (Mason, 2002; NRC, 2000; Sherin et al., 2011). Gamoran Sherin and Van Es (2009) emphasised that to use student thinking in instruction, teachers need to "learn to notice", and this was achieved when teachers critically attended to students' mathematical thinking while watching videos and then being more responsive to student thinking in their instruction and learning from students while teaching.

Studying Teaching in an LC

Recent large-scale studies by Popova et al. (2021) and Wei et al. (2023) identify "lesson planning and enactment" as key features of effective teacher professional development.This aligns with earlier work by researchers such as Morris and Hiebert (2011), who emphasized that making teaching practice accessible for collective viewing, discussion, and analysis is crucial for improving instruction and supporting teacher learning. However, classroom teaching continues to be a largely private endeavour and capturing high-quality records of practice – including classroom videos and other lesson artefacts – in the context of K-12 classrooms is a challenge. The LC is a setting designed for collective observation, study, and analysis of teaching and learning. The idea of LC for learning to teach mathematics is not common but is also not new. Under Dewey's guidance, the University of Chicago ran a laboratory school from 1896 to 1904. This laboratory school had two purposes:

(I) To comprehend and critically analyze theories about teaching and

(II) To develop new principles and domain-specific teaching theories.

One can see that the purposes are similar to any other laboratory in general. However, the basic principle necessarily demanded a very considerable break with the aims, methods, and materials familiar in the traditional school. It involved a departure from the conception that education materials and methods are already well known and need only to be furthered, refined, and extended. It implied continuous experimentation to discover the conditions under which educational growth occurs. Berliner (1986) and Schoenfeld (1988) also elaborated on how teachers learn from the practice of experts, which calls for genuine teacher education laboratories, where teachers, specifically novice teachers, would study pieces of experts' practice in order to learn to teach. In many other professions, like medicine or law, practice is developed in real and laboratory settings. The LC is one example of an effort to develop a site for experimentation and study intervention in mathematics teaching. The laboratory setting explicates the practice of "public teaching" (Ball, 2014), where classrooms are crafted for the collective observation, study, and analysis of teaching and learning. The conceptualization of public teaching as making practice visible to others requires the public teacher to play a dual role as both the actual teacher of the students in the class being studied and a "teacher" of the observers. Thus, the LC are settings in which the often private, partially invisible, and highly complex teaching practices are made visible for observers to study and investigate (Ghousseini & Sleep, 2011).

This study situates the laboratory as a setting for professional development to learn specific teaching practices, facilitating opportunities to enact and study practice in the laboratory.

METHODS

The paper analyses teachers' participation in an LC for one week as part of the summer mathematics program in a school of education at a mid-west University in the USA. The following sections describe the laboratory setting, study participants, data sources and methods used for the data analysis.

Setting of the LC

The laboratory consisted of a mathematics class for entering fifth graders who have not experienced success with mathematics in school. The mathematics class was collectively planned and studied by a diverse group of professionals, including teachers, researchers, teacher educators, student teachers, and mathematicians. The learning opportunities for teachers were organized in three events in the LC:

(1) A pre-briefing,

(2) Class observation, and

(3) A de-briefing.

Pre-briefing and lesson plans

The pre-briefing sessions happened before the actual enactment of the lesson. During these sessions, the LC teacher (the teacher who taught 5th graders in the LC) presented the day's lesson plan, explained the goal and activities for the class, and raised any concerns she and other lesson planners had about the lesson or particular students in the class. These sessions lasted for 60 minutes.

Special lesson plans for visible teaching

The lesson plans prepared for teaching in the LC were more detailed than typical lesson plans. They attempted to make the teaching in the LC visible to observers and to ensure that all possible student responses/ classroom scenarios were anticipated and considered before actual instruction.

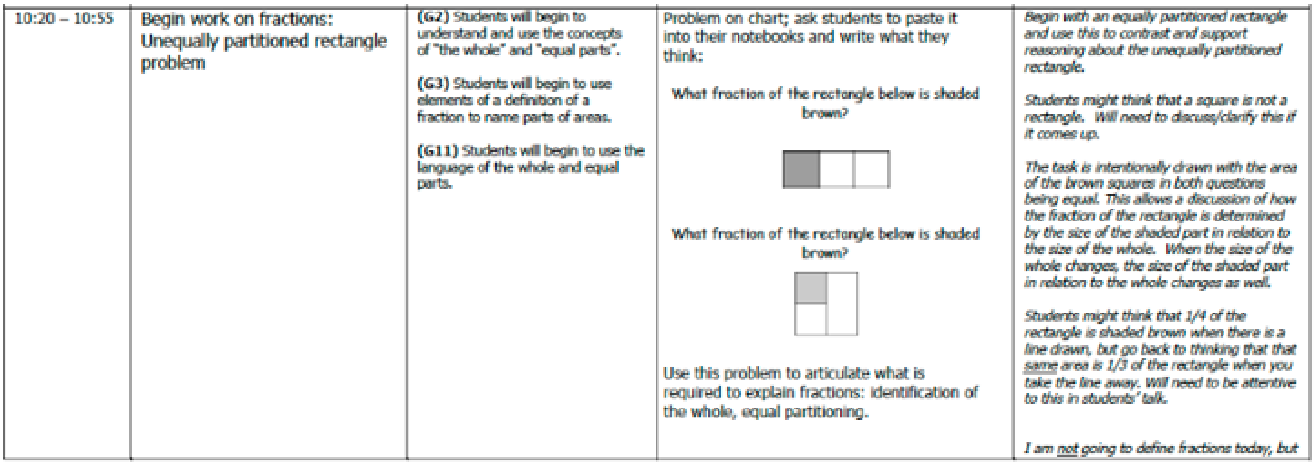
See **Figure 1**, which shows a chunk of teaching on identifying unequally partitioned fractions. See the right-most column; it describes the approach to teaching and lists possible student thoughts for the shaded area that looks the same across both figures yet is different. It discusses how unequal partitioning is a challenge with other considerations, such as square and rectangle relations. Participant teachers also used the lesson plans to ask questions about particular parts of teaching during de-brief sessions, to take notes and obtain a holistic picture of the lesson.

Class enactment and observation

The LC teacher taught the class, and other participant teachers observed the enactment. During the two-and-a-half-hour instructional period, observers were seated on risers in the back of the LC. Students were made aware of these participant observers, and participant observers signed and followed consent not to disturb the practice of teaching through any act.

De-briefing and planning for the next day

After students left the classroom, the participant teachers were invited to study the students' work in their notebooks and on the whiteboards. Post-which, a de-brief session was hosted. Participants shared their analysis of the enacted classroom and gave suggestions or anticipation for the next day's teaching. The participant teachers spoke most in this session compared to the pre-brief and asked questions to the laboratory teacher. These discussions lasted for 90 minutes every day.



10:20 – 10:55	Begin work on fractions: Unequally partitioned rectangle problem	(G2) Students will begin to understand and use the concepts of "the whole" and "equal parts". (G3) Students will begin to use elements of a definition of a fraction to name parts of areas. (G11) Students will begin to use the language of the whole and equal parts.	Problem on chart; ask students to paste it into their notebooks and write what they think: What fraction of the rectangle below is shaded brown? What fraction of the rectangle below is shaded brown? Use this problem to articulate what is required to explain fractions: identification of the whole, equal partitioning.	*Begin with an equally partitioned rectangle and use this to contrast and support reasoning about the unequally partitioned rectangle.* *Students might think that a square is not a rectangle. Will need to discuss/clarify this if it comes up.* *The task is intentionally drawn with the area of the brown squares in both questions being equal. This allows a discussion of how the fraction of the rectangle is determined by the size of the shaded part in relation to the size of the whole. When the size of the whole changes, the size of the shaded part in relation to the whole changes as well.* *Students might think that 1/4 of the rectangle is shaded brown when there is a line drawn, but go back to thinking that that same area is 1/3 of the rectangle when you take the line away. Will need to be attentive to this in students' talk.* *I am not going to define fractions today, but*

Figure 1. Section of the detailed lesson plan; the last column shows anticipation around tasks of teaching (Part of the lesson plan - Co-planned with the Lab Teacher)

Table 1. Participants' profile

Sou	No	Range of teaching	Teaching experience
Maths teacher	20	2nd to 6th grade	7 with < 2 years; 13 with > 4 years
Maths resource teacher	1	2nd to 6th grade	> 5 years
Retired maths teacher	4	3rd to 7th grade	>10 years
Maths teacher educator	1	Pre-service teachers	< 2 years
School principal	2	2nd to 6th grade	> 7 years
Math-education researcher	3	2nd to undergraduate	> 3 years

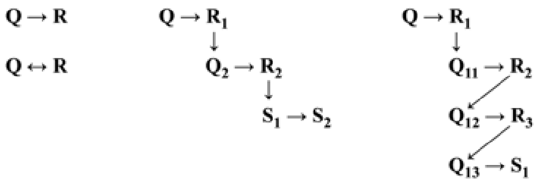


Figure 2. Structure of the idea unit in teacher discussion (Q - question, R - response; S - suggestion; First suffix indicate the order of question and the second designate teacher) (Source: Authors' own elaboration)

Mathematical content of the LC

The topics and practices were determined based on the difficulties students and teachers face while learning and teaching middle school mathematics. Understanding elementary fractions, learning to reason and justify ideas, learning mathematical vocabulary, and preparing a more substantial number sense comprised some of the mathematical objectives of the LC.

Participant Teachers in the LC

The study reports discussions among 31 math educators and researchers. Only one teacher has taught mathematics for less than a year. Each has taught various school grades, ranging from second to sixth standard. **Table 1** presents the composition of the group who attended the LC.

Data Sources

Video records of the participating teachers' discussion on the classroom enactment and their suggestions on the lesson plans were transcribed for all days of the lab class. Every day, the teachers discussed the classroom enactment for about 90 minutes in the de-brief session, and they spent about 60 minutes reading, editing and discussing the lesson plan of each day in the pre-brief of the lab class. In total, teachers collaboratively worked for 15 hours analysing and planning the nine-hour instruction for 5th graders that was spread over the six days of the LC.

Data Handling and Analysis

Specific "idea units" (Jacobs & Morita, 2002) were identified from the transcription of pre- and de-brief sessions to highlight teacher noticing. An "idea unit" is a segment discussing one particular idea/topic.

Each idea unit, in the case of discussions around the LC, consists of a primary question (Q) – almost always asked by the observing teacher, which generally begins the discussion of that idea or topic, followed by response/s by the laboratory teacher (R1, R2) with sometimes more clarifying questions from other participant teachers (Q1, Q2, etc.). **Figure 2** summarizes the structure of idea units present in the discussion in LC.

Table 2. Investigating professional vision (PV) in the discussions around the LC enactment

Processes for PV	Dimension analysis	Coding categories
Selective attention	Actor	Student – LC / school
		Teacher – LC
		Curriculum
		Self – as a teacher in the school
	Topic	Classroom management / culture
		Language aspect of teaching mathematics
		Students' mathematical thinking, (knowledge, reasoning, errors)
		Assessing students learning
		Pedagogical decision making
Knowledge-based reasoning	Stance	Description
		Evaluation
		Interpret
	Students' mathematical thinking	Re-stating students' ideas
		Unpacking the mathematical thinking
		Identify mathematical preparation
	Evidence	LC
		Personal teaching
		Other

The most common structure of the idea units observed where response from the laboratory teacher either prompted sub-questions (Q11, Q12, etc., second subscript indicates that the teacher is different) or prompted the teachers to provide suggestions for the next days' teaching (S1, S2, etc.). These suggestions were rarely challenged and often complimented by other participant teachers. The suggestions offered in the discussion were often incorporated in the next day's teaching. The results section discusses some annotations of teacher discussion that specifically highlight this feature of the LC – how teachers' suggestions were actually tried in the LC teaching and how that brought a greater authenticity to what teachers' decided to suggest.

Analyzing change in the professional vision

With the assumption that teachers benefit from opportunities to reflect on teaching with authentic representations of practice, observing live teaching in a laboratory setting seems to be providing a window into the classroom that conveys "the complexity and subtlety of classroom teaching as it occurs in real time" (Brophy, 2004, p. 287). Moreover, the laboratory design facilitated collaborative lesson planning and investigation that might have also helped the teachers to study the practice as a community. However, to understand and track what actually changes when teachers participate in the LC, the lens of professional vision is used in this study. Building on Goodwin (1994) and Sherin (2001), the definition of professional vision for the analysis of teachers' participation and learning is operationalized as follows:

> Professional vision consists of the ways of seeing and understanding events of the classroom that are closely tied around unpacking students' mathematical ideas, thinking of representations and examples to improve upon students' thinking and understanding mathematical preparation required for teaching that provides space for the students' mathematical ideas.

Based on the definition above and building on the coding scheme that Sherin and Vas (2009) used in the study of teachers learning in a video club setting, the idea units presented in the teachers' discussion were analyzed for two things – teachers' selective attention and their knowledge-based reasoning. Teachers' selective attention attempts to capture what is the focus of teachers' noticing. It included – whom they noticed (actor) and what they noticed (topic). They noticed or chose to talk about included the LC teacher, the student in the LC, students from their own classrooms, curriculum used or themselves as a teacher in their school. The topics that were noticed and discussed came from a wide range of teaching concerns, falling into five prominent categories – classroom management/ culture, language aspects of teaching mathematics, students' mathematical thinking, assessment and pedagogical decision-making. Idea units coded as classroom management included discussions on managing time, material or particular student behavior, whereas culture focused on discussions around developing a social environment conducive for the learning. **Table 2** provides the list of all the codes that were applied to each turn of talk in every idea unit. It is important to note that in each idea unit the topic or idea discussed remained same (see definition of the idea units) throughout all the turns of talks it comprised. However, the actor, the stance, the evidence, etc. differed and therefore documented for each turn of talk.

To examine the nature of the teachers' knowledge-based reasoning, each turn of talk within an idea unit was coded in terms of the teachers' general approach for making sense of the issue under discussion. It is called "stance"; the stance that the teacher used to discuss the observed teaching, namely, describe, evaluate, or interpret. In a describing stance, a teacher would offer an account focused on observable features of the teaching activity; evaluating included judgments about the quality of the interactions or decisions made by the laboratory teacher; and interpreting included inferences about what took place in the LC. In addition, the turns of talks in which the actor was coded as student and the topic was coded as students' mathematical thinking, a sub-code was added to strategize the patterns in the ways teachers talked about students' thinking. Three strategies that the teachers used to reason about students' ideas were:

1) Restate student ideas (i.e., quote/re-phrase what a student said),
2) Unpack the mathematical thinking (i.e. elaborate on the mathematical work behind what the student said), and
3) Identify mathematical preparation (i.e. based on students' thinking, discuss on kinds of mathematical preparation is needed).

The third component of knowledge-based reasoning refers to those turns of talks where teachers provided an explicit reference for the basis of what they were saying and this was coded as evidence. LC was coded as evidence if something happened in the teaching that they observed was the basis for their comment/question and when they used their own school experiences to support their comments, it was coded as personal teaching. To get a better handle on what these codes mean we code a section of an idea unit in the following paragraph.

Following is a section of an idea-unit that discussed students' work on a grid rectangle activity.

Participant teacher: I have a question about how you were scaffolding the language about how to record the grid rectangles, and I was just curious to know have you used the word, rows and columns, have you chosen, not to use them, because they get confusing, because...

LC teacher: Umm... Since I am not thinking about these as arrays, I mean they are in a way. But I am focusing more on the rectangle part of them and not the arrays. They are rectangular arrays, I know, but I am focusing on rectangles. I actually didn't want to get into that, as much. But I don't know whether I am right about that. I was thinking more about whether to teach sort of 'how high' with it. It becomes a different thing, because then you really are just counting. I don't know, what do you think?

Participant teacher: I think, I have taught the lesson, over the years and sort of fumbled with the language, when I taught it, it was a Math Trailblazer lesson. They used the language of arrays, as rows by columns; I am not sure whether it made it easier. So, I thought that it was interesting that you were asking how high it is and writing that number first. So, I was just paying attention to that and I think I like that better.

(Day 3, LC-debrief session)

In the discussion above, the teachers are referring to an activity around the grid rectangle, in which students were asked to generate rectangles for given numbers. For example, for number 6 students created a rectangle of dimension 3 by 2 using grid-rectangles, and then for each number they were generating as many possible rectangles as they could. Therefore for number 6, they would make two rectangles – 2 by 3 and 6 by 1. The LC teacher concluded the activity with recording students' responses for each number. The discussion above refers to the language nuances and complexity during that particular activity, especially when the LC teacher documented students' answers, dimensions of the rectangles on the board. The discussion further moved on to figuring out the conventional use of array language in higher mathematics, particularly in graphs. A concluding comment from one of the participant teachers given below also illustrates how idea units were often ended with participant teachers either suggesting a change or activity or demonstrating the understanding they gained during the discussion.

Participant teacher: … Also when we are talking about the rows and columns, I have taught eighth graders for the last ten years. And they still... because when we worked with length and width, then now we are switching it again. Because then it becomes your umm… column times your rows, whenever you are looking at, length and breadth, And seeing this being taught at this very level, really helps me to understand, some of the things I worked with the middle school and by the time kids get to eighth grade, if they haven't got this down, I understand why they are so confused, it just makes more sense about what's going on in their head. Because of the different things – are you talking about x, y, or are you talking about columns and rows, are you talking about length and width and then by the time you are trying to get them ready for the high school, and they still haven't kind of figured out, this is just, its really get jumbled. So, this is really making something for me, for my middle school teachers as to what's going on.

Table 3. Codes given for knowledge-based reasoning and selective attention

Transcript number	Actor	Topic	Stance	Mathematical thinking	Evidence
[PT$_a$1]	Laboratory teacher	Language aspects of maths teaching (language for grid rectangles)	Descriptive (how you were scaffolding the language…) Interpretive (have you chosen not to use them, because they get confusing		
[LCT1]	Self curriculum	Language aspect of maths teaching (didn't want to get into that… was thinking more about… sort of "how high")	Descriptive (describing what was the focus) Evaluative (don't know whether I am right)		
[PT$_a$2]	Self curriculum students	Language aspects of maths teaching (it was interesting how high it is and writing that number first)	Descriptive (description of self-teaching) Evaluative (it was interesting how you used how high…)	Rephrasing – students' thinking (language of arrays as rows by columns – not sure whether it made easier)	Personal teaching
[PT$_b$3]	Students	Language aspect of maths teaching (different words for arrays at different levels)	Descriptive (description of self-teaching) Interpretive (and seeing this taught at this level really helps me; so this is really something making for me)	Unpacking students' confusion (I understand why are they confused…because of the different things … its really get jumbled)	Laboratory setting (this is (the discussion in de-brief) is really making something for my middle school teachers – as what's going on)

Table 4. Characteristics of teachers' participation in the discussion around the live teaching observed

Participation characteristics	Meaning and codes generated
Positioning	Indicates how teachers referred to themselves and others during the conversations (participant teacher, co-planner, a mathematics teacher, etc.)
Proposing improvements in teaching	Passive or active. Passive is where they gave advice in general (e.g., actual visuals of the fractions would be useful) and active is when they had actual enactment in their mind (describing the complete making and use of activity or teaching aid)

The comment above brings forward a fine distinction between conventions that we teach in the school and its contextual usage during vertical mathematization (Freudenthal, 1932) in the school mathematics. It highlights a conflict in what language and practices students use in earlier grades and how those are different in their advanced education, which are almost presented as a new learning content. The participant teacher admits making sense of different encounters of the same mathematical expression as rows × columns, x × y and length × breadth, etc., as the root for causing confusion in students' minds in learning arrays.

Table 3 shows how turns of talks within this idea unit were coded. The idea unit is on "language aspects of teaching mathematics" and every turn of talk within it is coded.

Analyzing change in teachers' participation

Lave and Wenger (1991) argue that learning occurs as one participates in the community of practice through a process called Legitimate Peripheral Participation (LPP). LPP suggests that the mastery of knowledge and skill requires newcomers to move toward full participation in the socio-cultural practices of the community. LPP is not a teaching technique but an analytical viewpoint on learning, a way of understanding learning, and therefore used for designing learning opportunities, individual's role in that opportunity, "viewing learning as a legitimate peripheral participation means that learning is not merely a condition for membership, but is an itself an evolving form of membership" (p. 53), emphasizes individual's participation in a community of practice and change in it, as an indication of learning from the community. The legitimacy of participation is about ways of belonging and therefore legitimacy is not a condition of learning but a constituent of content; whereas peripherality indicates that there are multiple, varied and inclusive ways of being located in the fields of participation defined by community.

To understand the collaborative inquiry (Smith & Bill, 2004) into practice, teachers' participation in the debrief and pre-brief discussions was analyzed, where learning is seen as evolution in the nature of participation. Learning in itself is an evolving form of membership, and therefore in this data analysis an attempt is made to understand the membership. This was studied by identifying two characteristics of the teachers' participation:

1) Positioning, and
2) Quality of suggestions for the teaching.

The analysis was done to see how these teachers position themselves in the discussion; what was their involvement in the discussion – whether they gave suggestions for teaching, or they provided evaluative comments on what they saw in the class. **Table 4** summarizes these analytical characteristics.

RESULTS AND DISCUSSION

The results are presented in order of the research questions. Answers to these questions also unravel the shift in participant teachers' professional vision. In particular, results illustrate a shift in these teachers' noticing, change in their knowledge-based reasoning and change in their participation over the period of the lab class. A shift or change is understood based on the change in number of occurrences for particular codes. For example, the occurrence of codes related to students, their ideas and thinking increased over time. The teachers' understanding of students' ideas also captured qualitatively through interpretations they provided. The shift is understood from what the teachers portrayed on the initial days. In each of the findings below the shift is explained through actual examples, and transition in codes.

Shifts in Teachers' Noticing

Many topics based on the teaching viewed were discussed by the teachers in the de-brief session. During pre-brief, the teachers and the laboratory teacher discussed the day's lesson plan and made some final suggestions for the teaching and in the de-brief, that happened after the classroom enactment, teachers posed various questions, suggestions and concerns for the teaching. **Table 5** provides the range of topics that teachers discussed, the number of total turns of talks, the turn of talks by the laboratory teacher and the participant teachers during the de-brief sessions for all the five days.

Each box in the table has names of the idea-units with the number of turns of talks it consisted of, given in a bracket. The bulleted list under each idea unit indicates the specific mathematical or pedagogical topics referred under that idea unit. Each vertical column represents the days in the LC. Day 1 had the maximum number of idea-units and therefore maximum number of turns of talks. Maximum numbers of idea units indicate shorter discussion on varied topics. Days 4 and 5, although had a smaller number of turns of talks, they had longer turns of talks. Not necessarily every idea unit that appeared on day 1 appeared every day after that. Some of the consistent threads through all the days in the discussion involved idea units on language aspects of teaching mathematics, on students' thinking and ideas, on teaching styles and on material making and using. There were some themes that appeared only on the initial days. Some of those were homework, assessment, how to handle students' errors and making and using notebooks.

The threads that reoccurred on the other days, has some element of progression in it. For example, managing the classroom and students was discussed on three of the days, with a progression in the discussion that lacked first and then addressed the students' view

Table 5. Various topics discussed: ToT = Turn of Talks, LT = Laboratory Teacher, PT = Participant Teacher, IU = Idea units

Day 1 (ToT = 131; LT = 48, PT = 83) IU =30	Day 2 (ToT = 118; LT = 45, PT = 73, IU = 22	Day 3 (ToT = 115; LT = 46, PT = 69, IU = 15)	Day 4 (ToT = 76; LT = 23, PT = 53, IU = 10)	Day 5 (ToT = 63; LT = 17, PT = 46, IU = 11)
(48) Discussing language aspect of teaching mathematics · Around fractions (13) · Around using arrays (27) · Of the questions posed (2) · Of definitions (1) · Around misconceptions (4) · Conventions (1)	(22) Discussing language aspect of teaching mathematics · Of the questions posed (10) · Around students' terminology (4) In introducing new mathematical terms (8)	(32) Discussing language aspect of teaching mathematics · Around student terminology (18) · Of the questions posed (3) Students' language of fractions (11) – counter to the day 1	(21) Discussing language aspect of teaching mathematics · Students' language of equality (18) · Reading formal instruction – naming fractions (3)	(9) Discussing language aspect of teaching mathematics · Compressed and contextual terminology for students (6) · Modeling to help students speak to the class (3)
(28) Challenging mathematical topics for students – errors they make · Equality (2) · Infinity (3) · Fractions knowledge (23) -Choosing a whole (8) -Distinguishing shaded and non-shaded (7) -Ratio Vs. fraction meaning (1) -Fractions meaning (1) -Equal Vs. non-equal parts (1) -Equal parts in early grades (5)	(22) Discussing students' thinking and ideas about · Equipartitioning and naming fractions (11) · Number sentences (3) · Equation writing stereotypes (4) · Prior understanding of rotation and symmetry (4)	(22) Discussing students' thinking and ideas · Articulation of the ideas (7) · Verbal articulation Vs. written articulation of thinking (3) · Unpacking individual students thinking – analyzing to respect others and reflect on one's own (12)	(36) Discussing students' thinking and ideas · Modeling writing for students' thinking (12) · Unpacking individual students' thinking (6) · Unpacking students' thinking about recording their own work (5) · Fixing rectangles to see pattern for prime numbers (work of two students) (6) Eliciting students' thinking in the classroom (7)	(32) Discussing students' thinking and ideas · Organizing notebooks to track students' thinking (6) · Depth in students' reasoning (11) · Students' thinking as peer assessment (9) · Unpacking individual students' thinking (6)
(19) Assessment · Pre-assessment (11) · Individual – informal assessment (4) ·End of check (2) Notebook (2)	(20) Materials making/quality · Making templates for fractions (6) · Using Cut-outs to cover fractions (10) · Animated video (4)	(12) Materials/tasks making/quality · Material in anticipation (7) Questions based on students anticipated responses (5)	(7) Materials making/quality · Making use of rectangle grids for various purposes (7)	(10) Materials making/quality · Making use of rectangle grids for fractions purposes (6) · Visual representation of square root (4)
(9) Classroom Management · Serious teaching (2) · Activity method (2) · Setting up white boards (2) Using lesson plan (3)	(19) Classroom management · Managing a student (15) · Managing classroom during hands-on activities (3) · Maintaining conducive environment (1)	(14) Deciding students' Engagement · Listening (3) · Writing in the notebook (3) · Speaking in the class (8)		
(6) Homework: What's in it (6)	(13) Strategies for teaching fractions · Hands-on experiences (7) · Practice-practice (3) · Creating conflict in what students know (2) Open ended tasks (1)	(17) Strategies to facilitate learning · Introduce a conflict-analyzing errors (12) Exploring what students can do (5)	(9) Strategies to facilitate learning · Using an assistant teacher in the classroom (4) · Using a fish bowl (5)	(12) Teaching comments · Amount got taught in the lab class is of 2 weeks in the school (7) · Problem solving teaching style (5)
(4) Notebooks · What to write in it (2) Making use of pens (2)	(6) Indicators of learning · Giving explanations (4) · Giving generalizations (2)	(18) Indicators of learning · Time spending (8) · Sets of milestones/ pointers/ questions (10)		
(12) Handling student errors · Creating a safe place (4) · Impact on others (2) Keeping track of errors (6)	(10) Handling student errors · Giving student a benefit of doubt (8) · Correcting the error (2)			
(5) Other · Student contract (4) Wrap-up (1)	(6) Making teaching visible · For professional development (6)			(3) Co-planning as learning to teach together

in it. On day 1, discussion on classroom management occurred in the context of styles of teaching. A "serious" teaching – i.e. the teaching without much humor and the one that sticks to mathematical discussions and an "activity-based" teaching; both were discussed for its function in classroom management. On day 2, the discussions on management concerns were rooted in a context around a particular student's behavior, which was then generalized to ways for managing "difficult kids" in the classroom. This particular concern is discussed in detail further in results where one of the teachers speaks about teaching practices that facilitate eliciting students' knowledge in the classroom setting and hence resolving the management concerns. However, on day 3, a profound discussion on understanding students' engagement took place. The teachers discussed how most classroom management decisions are made based on students' behaviors (e.g. where are they looking at) rather than actually finding out what they are listening to. During this discussion the teachers brainstormed, associating students' engagement with listening meaningfully in the class, writing in the notebook and students' speaking in the class rather than just focusing on how much the student is looking at the teacher.

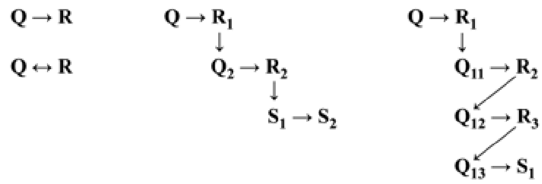


Figure 3. Structure of the idea unit in teacher discussion (Q - question, R - response; S - suggestion; First suffix indicate the order of question and the second designate teacher) (Source: Authors' own elaboration)

The discussion around language aspects of mathematics teaching also progressed in a meaningful direction. This particular thread has more turns of talks. Beginning with topics such as, how students were not using any formal mathematical terms and how that is problematic for learning on day 1, the discussion moved on to understand the terminology that student brought in the classroom. The language discussion shifted from what students didn't know to what they know. The teachers pointed out the students' terminology; ½ of ¼ or ½ of ½ [half of quarter or half of half] making proper conceptual sense, even though there is no reference to $1/8^{th}$ or $1/4^{th}$.

Similarly, discussion on the student ideas shifted from just listing what errors students did to eliciting the thinking that underwent those errors. Since day 1, the teachers noticed how the laboratory teacher is facilitating students' thinking in the classroom. However, the change was how they understood the role of students' thinking in the classroom teaching. Early on, the teachers gave little attention to students' mathematical thinking, even when prompted to do so by the laboratory teacher. The discussion disregarded the unpacking, i.e. finding out the systematic thinking (if any) that led to the error and only focused on errors made by the students. Following is an example to elaborate this shift.

On the first day of the discussion, the teachers' focused on, how lack of "clear answers", "clear instruction" or "knowledge of mathematics procedures" could lead to errors in students' work, where as, towards the end the focus was more on unpacking what is students' reasoning irrespective of whether it led to right or wrong answer.

For example, on day 1 there was a problem posed for students to name fraction for the shaded part of the given figure, where a rectangle was unequally divided (see **Figure 3**). Students came up with three different answers for the problem – 1/3, ¼ and 1 ½. For every student's response the LC teacher probed other students to ask questions till they understood what is the solution and how it is obtained. The student with answer 1 and ½ justified that the big rectangle (half of the square) in the figure is a whole and therefore the non-shaded portion of the figure is 1 and ½. After her explanation to the whole class, in the next round of naming fractions exercise, many students followed the approach described by the student. They all took liberty in deciding the whole and then finding a name for the shaded part. It was apparent that students were influenced by the strategy of choosing the whole on their own and however they all were mathematically correct, they arrived at different names for the same shaded fraction.

During debrief of this class, teachers questioned the promotion of the idea of choosing different wholes, disregarding the mathematical importance of it and only focusing on how it might lead to different answers.

> PT9: I loved when you were getting their guesses, their educated guesses about the second picture, where it was divided into three unequal parts, you listed their three guesses and one child had 1/3rd, one with 1/4th and other came up with 1 and a half. And she was the only one admitted publicly that getting that answer and no one else admitted that and there is little bit of discussion of how she got that and which I kept thinking that wow she is great, well when we looked at the journals what happened is several of the students convinced themselves that her method is the right method instead of a wrong method.

The reference to student's method as a wrong method indicates that idea of flexible wholes is taken as a problem for teaching rather as an opportunity to develop real and complex understanding of what is a whole in relation to naming fractions. Naming of fractions is dependent on the size of the whole, and therefore idea of flexible wholes is not only exercising what fraction means, but it also unpacks a critical component of the process of naming fractions, i.e., deciding the whole. When asked by the LC teacher whether it is really wrong, there were few turns of the discussion among teachers.

> PT16: Well, it is not wrong but again I keep going back to my thought that it is genius, you put it so eloquently that she pursued the whole and that's how she got one and a half, but my question is… my observation is now what we do tomorrow, with all of these other kids, who have convinced themselves, that that is how we come up with answer?
>
> (Participant teacher in the LC—debrief)

The major chunk of the discussion was taken up by these questions, where they discussed whether choosing wholes to name fractions is wrong in itself and what they should tell students in order to teach the naming of fractions. Another objection was that the fractions 1 and ½ was used to name the un-shaded part, even though the question specifically said shaded part. Teachers' concern for one definite answer seems to be rising from the tests that students have to undergo.

> PT7: I don't know whether I need a microphone, but ... same thinking and deliberately teach shaded and un-shaded, and I am wondering if that would have…hmm... less confusing for that student, as she brought these terms up… Because of the MEAP test, they will try to trick the students with shaded and then un-shaded.

However, in the LC room, repeated exposure to students' thinking, made teachers notice students' explanations and not just their right answers. On day 3 one of the participant teachers is reflecting on students' oral explanations and hinting that similar needs to be worked out for their written work.

PT23: One thing I noticed with Alison, at the beginning of the week I saw explanations being not that good, like "because I got the right answer", but I am starting to see that she is been able to articulate a little bit clearly and why she did what she did. Not just saying that she got the right answer but she might say something like, both sides are equal or she will be more specific, if this side is 5 then the other side needs to be 3 for number 15. So I think their explanations for the majority have gotten clearer. I will probably still focus on that in their notebooks…

The teachers not only began to notice students' explanations but also discussed how eliciting those or anticipating those is part of the work of teaching.

PT1: Working on errors not only creates safety but also respect for each other's thinking, like you would listen to each other when you know there is thinking behind it.

LT: Schooling is the place to make people think and develop civility about others' thinking. It is not only math but also any subject, literacy, social sciences, that requires developing an attitude that teaches people to listen to others' reasoning and thoughts.

PT2: The task that involves explaining why is the answer wrong or right, or what is the reasoning demonstrates that the teacher herself knew that kids have these different kinds of reasoning.

The teachers' encounter with students' thinking in the classroom observations and engagement with the students' work in the notebooks, on the charts, changed their perception about students' thinking. Towards the end of the LC there was a sense that something like "kids' mathematics" exists and they as teachers need to take that into account. Following is a quote, from a teacher that indicates awareness of what s/he needs to know as a teacher about the students. Further, there seems to be a conjecture that creating space for students' thinking/mathematics in the classroom reduces the behavioral management issues.

PT21: …I wanted to say and I said this to somebody else but, it was a rough year for me, this past year, we had just a lot of discipline problems, and coming this week, when I ended the year I felt like, I don't know whether I can go back to school next year. And, coming here, just kind of made me... yeah... I can do it, I feel a lot better now, and I realized why I like this age of students.

LT: What was about the week that affected how you felt last year?

PT21: Well, they are just not all bad...I don't know, I hear people comment that some years you just have a year. And this year, at our school at least, it wasn't just my class, it was all three classes, almost everyone that came in contact, with the teacher of fourth grade said, oh my goodness, good luck, 5th grade (everyone laughs). But now I feel that may be we are not taking the kids seriously… there is so much mathematics that they do…

PT11: We had the same thing, with some of the students, they are here in the class [LC] … they are some of our more problem students, so it was interesting to see them so engaged.

The teacher, who shared her feelings of not wanting to go back to the school, was emotional when she spoke. She was striving to teach better, but discipline issues didn't let her. However, her noticing, "there is so much mathematics they [kids] do" and "we need to take the kids seriously" manifest a shift in understanding what the kids are doing in the classroom. These specific comments by the teachers above also respond to the classic dilemma that teachers' face, choosing between moral and intelligence (Ball, 1993). The kids' mathematics that the teacher is referring to in the quote above, would be part of the wrong mathematics in her classroom, making it morally difficult for her, as the work of teaching then would only entail correcting the students' mathematics. However, after viewing what students did when they were given a chance to build upon each other's understanding, PT21 is conjecturing that maybe if she could create more space for students' mathematics then the teaching won't be just the correcting the students' answers but needs to include building upon those. There is a hint of realization in her quote that making the reasoning behind the answer as main mathematics (instead of the answer), opens up opportunities for students to engage and teachers to let those students be part of the classroom teaching. There is more that needs to be understood in this regard, but still, making space for unpacking students' thinking, knowing and understanding it, appears to be the essential step for moving forward.

Table 6. Idea units in percent for knowledge based reasoning

			Day 1	Day 2	Day 3	Day 4	Day 5
Knowledge-based reasoning	Stance	Description	80.65	63.64	80.00	60.00	72.73
		Evaluation	54.84	36.36	06.67	00.00	00.00
		Interpret	12.90	27.27	80.00	60.00	63.64
		Re-stating ideas	09.68	27.27	60.00	80.00	72.73
		Unpacking the mathematical thinking	00.00	09.09	40.00	60.00	72.73
		Identify mathematical preparation	00.00	13.64	46.67	60.00	54.55
		LC	03.23	13.64	46.67	80.00	63.64
		Personal teaching	54.84	36.36	00.00	00.00	36.36

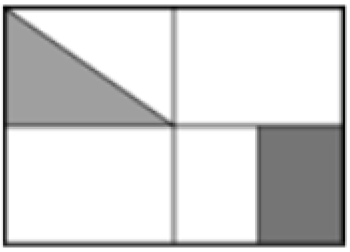


Figure 4. Identify and name the triangular and the rectangular shaded part of the rectangle (Part of the lesson plan - Co-planned with the Lab Teacher)

Changes in Teachers' Knowledge–Based Reasoning

The reference to students' thinking was elevated over time during the de-brief discussions. Teachers' stance on students' thinking was moved from being evaluative to interpretive. **Table 6** summarizes the number of idea units spent on unpacking and understanding students' thinking. Almost all the interpretive or evaluative stances were taken after a description of the phenomenon or story or beliefs or assumptions. There was less and less of an evaluation stance (54% to 0%) and significant rise in interpreting stance. The interpretations were often coupled with the suggestions for the next day's teaching.

Students' ideas were referred from day one, but its unpacking, mathematical relevance and therefore required preparation was discussed mainly from day 3 onwards (> 40%). The teachers provided evidence for their argument (see coding **Table 3**). On days 3 and 4, the evidence came from their observations and not from personal teaching. However, on the last day, the teachers reflected on their learning in these 5 days and spoke about how it applies to certain situations in their own teaching experiences. Re-stating students' ideas and unpacking the reasoning behind it took place often as teachers engrossed in their collaborative investigation. Often what was seen as an error was understood as systematic thinking and further attempts were made to figure out what the thinking was and what implications it has for future mathematical pedagogy.

The teachers discussed students' thinking on problems at length and others often supplemented what they learned from their extra investigation. Following are some discussion excerpts for students work on the problem shown in **Figure 4**.

PT7: I think for that particular activity, what I noticed in the notebooks, it just shows, where was thinking at that time, Janice, she really stuck out, because, she recognized it was one half of one fourth. That was her answer for each one, and it really gave us some insight in how she was starting to make the connection, that it is one eighth. But she just doesn't know the terminology for one eighth at that time.

PT18: Just a comment about the one fourth, one half of one fourth, that reasoning is visible directly in the picture, you see, you have a criteria, because you already have equally partitioned, in four pieces and then equal partitioned one of the pieces in two halves, where as the one eighth answer requires, more abstract kind of imaginary lines of the drawing.

This conversation is important as both teachers are sensing the meaning of half of one fourth as said by Janice, however, PT18 is also making a connection that to see the shaded portion as $1/8^{th}$ is difficult as there is one extra step, which requires seeing 8 equal parts of the whole [making it difficult with uneven partitioned figure]. These teachers further realize that imagining 8 equal parts for both shapes of $1/8^{th}$ as given in the figure (see **Figure 4**) is difficult and they come up with ways in which students could proceed on doing that imagination through a hands-on activity.

PT11: Here is a suggestion, see if Janice understands that it is half of one fourth…(repeats the sentence twice) then to show her it is also 1/8th, we can make cut-outs of the green triangle and give those in ample, to see whether any of them want to use those to figure out how many of those actually fit in the whole rectangle. We can do this with the blue rectangle as well, in fact the small rectangle would be easier…

The teacher here is not only thinking of a representation to help the student in identifying fraction names, but she is making it a customized one, which is based on students' understanding and anticipating the difficulty that understanding might lead to. The suggestion of the cutouts of the green rectangle is specific to students' inability to see eight equal parts in that figure. This particular tool

Table 7. Teachers' planning together in the EML

Task observed during enactment	Pedagogical content knowledge
Charts are around the room with the rectangles from 1 – 36. Provide Grid Rectangle Table synthesizing the information about the rectangles: number, factors, # of grid rectangles. Leave a few of the rectangles blank in order to give students a chance to better understand what information is included in the chart and from where it comes. Compare Grid Rectangle Table to the posters hanging around the room. Work together as a class to discuss and fill in missing information. Numbers left unfilled: 10–4 factors from an even number 16–Square number (odd number of factors-5) 18–composite 6 factors 27–4 factors from an odd number 29–only 2 factors?? Look for other numbers that fit in these categories? Other patterns you notice?	Possibilities: 1. Numbers that have only 2 grid rectangles are prime numbers, 1×p, p × 1 2. The number 1 has only one rectangle 3. If N gets bigger, that does not mean that the number of rectangles gets bigger. 4. Numbers that have an odd number of grid rectangles are squares because one factor pair (n × n) cannot be rotated to create a complementary fact 5. There are 3 rectangles exactly when N is the square of a prime number. 6. Numbers having 3 or more factors are all considered composite numbers. Look for patterns of numbers that have a common number of factors. 7. There are four rectangles exactly when either N is a product of two different primes, or if N is the cube of a prime (like $8 = 2^3 = 2 \times 2 \times 2$, or $27 = 3^3$) The number of rectangles depends only on how many primes occur and how many times each of those primes occur, but not on what the primes are.

is not a teaching-aid which has come in the picture just to conduct an "activity-based" lesson, but more as a product of comprehending students' thinking and noticing what support structure could be provided to take the thinking forward.

The teachers' discovery of kids' mathematics and reasoning behind what students said, appears more significant in contrast to what they noticed on the first two days of the LC. To get an idea of teachers' evaluative stance with respect to students' thinking, following are some quotes from the first two days of the laboratory.

> PT7: Well, I thought like they may be were having a hard time communicating their ideas and may be they needed to have more clarification of the vocabulary, that would help them in their communication. They were not using terms numerator and denominator, may be you need to tell them those.

The question above was asked in connection to students naming a fraction wrongly. Firstly, teachers, in general, never re-stated students' responses. The re-stating was discussed on day two when the students in the lab class were often asked to re-state each other's responses. The comment also suggests the assumption that the concept of fractions is understood when the terminology around it is known, which we saw in the earlier discussion was not the case. Again in reference to "clarification", "telling" seems to be suggesting the work of teaching that PT21 found uncomfortable as a work of teaching – mainly as the only purpose it serves is correcting what students know and inhibits the construction of the responses. Following is another response on students' misconception and its relation to teaching in general.

> PT11: I was just wondering if you have like an ongoing list in your head, like, stop like, if you see it on more than one kids paper, like misconceptions that would make you actually make you halt and alter your direction in a lesson.

The teacher here seems to be suggesting that what is the degree of errors that the lab teacher allows in the classroom and when is the "halt" point. This comment suggests two things – one, that the teacher is uncomfortable with the idea that it is okay to make errors and wants to know what is the upper limit of this, when is the moment when the mistake is not acceptable. It is important to note that initially the teachers noticed the errors students did but not the ways the errors were discussed in the classroom. Second, it also means that following the approach where kids have their own space of doing mathematics; the teacher would end up teaching something else than what she already planned.

Pedagogical content knowledge

More attention towards students' thinking, kinds of representations, elaborations on what to teach next day and substantial discussion on how to deal with students' mathematical ideas, demonstrated teachers' pedagogical content knowledge around mathematical topics such as fractions, equality, number-sense in the LC pre and debrief sessions. The discussion around the topics was always situated either in the LC-teaching context or in the context of their own teaching. Often parallels were brought from what they do usually and what they see now and how they plan to change.

Table 7 is an example of such collective knowledge. Here teachers gathered the number of possibilities for discussion on grid rectangle activity. In the class enactment a detailed discussion happened on square numbers. The fourth possibility, that numbers that have an odd number of grid rectangles are going to be square numbers, prepared the LC teacher to build on students' observation for the number of factor pairs. In the class, when students' observed the odd number of rectangles, the LC teacher could prompt them to find a pair that is making a square. Also, it lead to discussion that as one of them is square, rotating the dimension will not produce a new rectangle and hence the odd number of rectangles.

This production of knowledge helped the laboratory teacher in the class, but it also had learning importance in the context of teachers.

> PT3: I also have a second point about what I gained from this week… is again I think I have said this several times… from just one lesson, the grids, the rectangular grids, the amount that we taught… just for that problem, even today prime numbers and square roots, that's to me, that is so powerful, and I think…I was talking to Eddie about this that it wasn't so much

Table 8. Characteristics of teachers' participation in the discussion around the live teaching observed

Participation characteristics	Meaning and codes generated
Positioning	Indicates how teachers referred to themselves and others during the conversations (participant teacher, co-planner, a mathematics teacher, etc.)
Proposing improvements in teaching	Passive or active. Passive is where they gave advice in general (e.g., actual visuals of the fractions would be useful) and active is when they had actual enactment in their mind (describing the complete making and use of activity or teaching aid)

about that they will remember it, but they will able to recall it and make associations better when this is brought to their attention again.

PT16: …the square roots, and having the kids see the squares, it kind of lead to the conclusions on their own, just brought me back to even when I was in school, okay, I learned what a square root was, but I never had a visual of having to make a square, I didn't have that. That would have helped so much. Meaning wise it gives so much to think about…I wonder we need to work together for other such concepts…

The two quotes above explicitly describe how they find these representations legitimate for students' learning. Being aware of the geometrical representation of square numbers or operational understanding of prime numbers, made the classroom not only interesting but also mathematically dense. Teachers' recognition that mathematical preparedness of such kind and lack of exposure to be able to do that, confirms a call for specialized content knowledge (Ball et al., 2008) needed for teaching mathematics. Again here, there is a hint in the quote that teachers need settings where they all "need to work together" on other such topics, confirming the need for such collaborative settings.

Shifts in Teachers' Participation

Teachers' participation characteristics such as, how they positioned themselves in the discussion; what was their involvement in the discussion – whether they gave suggestions in teaching or they often took evaluative stand; what and whether their participation indicate any shift over the time are captured through in the sense (See **Table 8**, for detailed meaning of the codes positioning and proposing improvements in teaching). Based on the nature of the discussions during de-brief and pre-brief sessions, following characteristics were observed for teachers' participation, about and of practice of teaching.

Positioning

Teachers positioned themselves in multiple identities during the de and pre-brief discussions. Their self-positioning was accessed through how they referred themselves, the LC teacher and other teachers. Specifically, on day 1 teachers referred to themselves as individuals and often suggested to the lab teacher as "you can do this or try that", etc. However from the second day onwards, teachers constantly referred to themselves as a part of the group and even suggestions for the teaching were given to the whole group and not just the lab teacher.

> …I thought at one point specially when they said, when you were trying to talk about the differences between the two shapes they were looking at to specify the equal parts. Just because from... they took the assessment in this spring I knew... at least the students I worked with had a difficult time for the visual representation of the fractions. And so...That is mainly why I would have probably, if it would have been the teacher, I would have done that language part today.
>
> (Day1, Participant teacher in the LC-de-brief)

> As you were doing the number problems, to the sentences, ten was always on the right. Do you think of flapping that tomorrow or in the future so that kids don't always get locked into that? I think you should try that…
>
> (Day1, participant teacher in the LC-de-brief)

The above two comments were made on day 1. Here it is clear that they are referring to themselves as an individual and the lab teacher as the one who needs to work on the teaching. At the end of the discussion on day 1, there were few suggestions for the next day's teaching and the lab teacher used all those in her teaching the next day. There was a sudden change in the way teachers referred to themselves and the lab teacher since then.

> Some students have problems with just visual representations, they need association with verbal explanation or at least with some terminology. So, if we tell them to say how many equal parts every time they write fraction for shaded parts, will help them.
>
> (Day 2, participant teacher in the LC-de-brief)

> I think we need to offer something that will pull some of these things out, to make them more clear. Offer some of the terms so we can use shaded—unshaded and so on some other unequal parts. That is what we need.
>
> (Day 3, participant teacher in the LC-de-brief)

The first comment above was given when there was a discussion about students' work on naming an unequally partitioned fraction. This comment is not just an acknowledgment of the problem but a suggestion to support what can be done. This teacher's sense of identity as a co-planner is leading to the reference as “we”. Including the LC teacher as part of the group of participant teachers in their reference began to happen from day 2. Day2 was the first time teachers’ suggestions were concretely present in the lesson plan that was enacted. Following are some responses from day 1, which showed a different positioning than what we saw in the discussion from day 2 onwards.

Although the transition from “I” and “you” to “we” sounds trivial, it did happen, strikingly from day 2 onwards. Then on, “we” took responsibility for every action that happened in the LC classroom. The few references to “I”s came when the teachers wanted to describe something that they did in their own teaching and again on the last day, during a discussion on what they take from the LC to their own classrooms. The referencing to oneself as “we” did impact teachers’ engagement, especially in terms of the improvements they suggested.

Proposing improvements in teaching

From day 2 onwards, teachers provided suggestions in teaching that were more detailed, and with considerations of how to actually use the suggestions in the class. Every day the suggestions teachers gave were discussed and weighed to understand its relevance and use in the students’ learning in the LCroom. These suggestions then were actually used by the LC teacher, which brought greater authenticity and accountability to their suggestions. This whole process impacted suggestions for planning, which over time became more and more concrete.

> For teaching equivalent fractions, I tried the activity of superimposing cut-outs with my kids. We can take transparent sheets and make fractions on them. Color the parts, as you will do on a normal paper. For example, 2/6 and 1/3, you make a cut-out for 2/6 by shading 2 parts out of 6 and then make cut-out of 1/3 by shading 1 part of 3…the whole has to be exactly the same while making the cut-outs. We can use different kinds of shading as later when we superimpose it will be visible, distinguishable… they understand why those fractions are equal.
>
> (Day 3, participant teacher in the LC-de-brief)

The actual quote is very long and the details provided for the suggestion of teaching equivalent fractions are much finer. These suggestions were much more explicit than teachers suggestions on the initial days which were like – “need more clarification”, “clear language”, “telling clearly”, “use hands-on methods” or “teach with activity-based methods”.

As mentioned earlier, the design of the LC creates a complex space for teachers to learn. Sites such as the LC allow duality of perspectives (Naik & Ball, 2012) – one, where teachers constantly refer to their own identity as practicing teachers and second, as a co-planner of the collaborative planning and investigation of the teaching. During these five days, the teachers pursued their own interests, their own challenges. And above all, even though the teachers were part of the work of teaching, the accountability of actual teaching was not on their individual shoulder. Therefore, along with duality, there was a space for individual autonomy.

CONCLUSION

This study reported on an analysis of teachers’ participation in observation and collective examination of teaching in a lab class, where the lab teacher, an experienced elementary teacher, taught 30 rising fifth graders, all morning every day for two weeks. The participant teachers’ discussion based on observations of the teaching and lesson plans was analyzed for noticing their knowledge-based reasoning and participation. **Table 5** illustrated the variety, range and recurrence of what teachers noticed during these discussions. They discussed various topics, such as language issues in learning mathematics, challenging mathematical topics in the curriculum, issues with classroom management, using and making records in notebooks and so on. Among others, students’ use of mathematical language seems to be the most prevalent in their noticing. However, progression was seen in what they noticed about it. In the beginning, the teachers noticed how students were not using any formal mathematical terms and how that is problematic for learning. However, towards the end they were seen making sense of the terminology that students brought in the classroom and accepting it as a pedagogical resource. In a way this discussion shifted from listing what students didn’t know to what they know. For example, towards the end, the teachers pointed out that students’ terminology, such as ½ of ¼ or ½ of ½ [half of quarter or half of half] made conceptual sense, which was considered as lack of understanding 1/8th or 1/4th on the first day. Building on similar instances, teachers developed the idea of “kids’ mathematics” – the mathematics/ language/ notations that students bring to the classroom as a resource that could be used to build students’ formal mathematical knowledge.

It was not only that the teachers noticed and accepted the students’ thinking more, but they also began to decipher students’ responses. Teachers’ attempts of unpacking students’ responses and figuring out mathematical preparations for it, increased from 0% to 72 and 54 percent respectively. More attention towards students’ thinking, representations, elaborations on what to teach next day and substantial discussion on how to deal with students’ mathematical ideas, led to rich resources, such as the one in **Table 7**. These pedagogical resources were the outcome of the collaborative efforts. Such efforts were also visible in the ways teachers positioned themselves during the discussion. The transition of referring to themselves from “I” to “we” represents their collaboration. To summarize the shift in teachers’ professional vision was observed when the topic of their discussion was moved from “the lab teacher” to “the students”; when their noticing moved from being evaluative to interpretive; and when their comprehension of students’ responses led to concepts such as the “kids’ mathematics” and collaborative pedagogical resources.

Achieving a shift in professional vision, so that teachers become sensitive to those classroom events that are significant in unpacking students’ thinking and identifying representations, mathematical preparations needed to teach, is an important goal of any teacher professional development. In this context, the study proposes two things, one, that a medium that allows teachers to develop “analytic-mind set” (Sherin, 2004), a mind-set where classroom interactions are examined without constraints of instruction, requires an active and passive engagement with teaching practice. Such that, even though teachers investigate real classroom incidents, they get more time for making sense of it and taking decisions than a real classroom. The second thing this study proposes is a requirement of urgency to use what teachers noticed in the teaching. The teachers in the study constantly worked towards developing concrete suggestions for the next day's teaching and therefore, collaboratively produced pedagogically and mathematically rich resources. These insights have implications to all kinds of practice-based learning environments. The study suggests that developing practice-based environments that enable active and passive engagement with practice and also develop an urgency to construct pedagogical tools is a way to move forward.

Said this, the study does not quite investigate the role that the laboratory teacher played in it. The LC teacher is part of all the discussions and collaborative investigations as other participating teachers. However she also plays extra roles, such as:

(1) Collaborate with teachers in planning and then, figure out ways of teaching that will do justice to the teachers' suggestions and still maintain the mathematical density of the instruction;
(2) Teach the students so that they will get maximum possible assistance to progress in their learning and therefore collaboratively imagine all possible hazards while teaching and be prepared for them; and
(3) Teach the students in a way that the work of teaching is accessible to the participant teachers who are observing the teaching, but make sure that it does not become a rhetorical burden for students, hence has to be done with most subtlety.

So, it is challenging to operationalize the role of the lab class teacher and requires more complicated understanding to actually replicate the idea.

Acknowledgement: The author thanks to three mentors who helped improve this work immensely: Prof Deborah Ball, Prof Hyman Bass from the University of Michigan, USA, and Prof K Subramaniam (Retired), HBCSE, India. The work was supported under a scholarship author received from Rackham Graduate School and appreciates the support.

Funding: No funding source is reported for this study.

Ethical statement: The author stated that the study was conducted in accordance with ethical principles for research in the social and human sciences and was approved by the Institutional Review Board – HSBS (IRB HSBS) (First approval date: 2 November 2016, university requires extension of the approval annually. Last approval received for annual extension date is 10 October 2025) as part of a continuing review process, under reference number HUM00108557 / CR00066688.

AI statement: The author stated that the study has utilized the unpaid version of Grammarly to review the text for spelling and grammatical accuracy. All sections, with the exception of direct teacher anecdotes, have been checked using Grammarly; the teacher quotes remain verbatim in accordance with transcription integrity. Additionally, the free version of ChatGPT was used to rephrase the first two paragraphs of the introduction section and first paragraph of the conclusion section to enhance clarity and polish while maintaining the original meaning. The author has carefully reviewed and edited all AI-assisted outputs to ensure accuracy, coherence, and alignment with the scholarly intent of the manuscript.

Declaration of interest: No conflict of interest is declared by the author.

Data sharing statement: Data supporting the findings and conclusions are available upon request from the author.

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