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Visualization and properties of boards and tiles: The case of the Padovan or Cordonier sequence

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ABSTRACT

The study of recurring numerical sequences usually involves a restricted context of circulation and communication of new ideas and abstract mathematical properties. On the other hand, authors of the history of mathematics books tend to emphasize, in an almost restrictive way, anecdotal curiosities about the Fibonacci sequence. Given this context, in this work, we find a proposal for exploring arithmetic and combinatorial properties using the notion of board and tiles, which correspond to properties of the Padovan or Cordonier numerical sequence. Through some elements, we will demonstrate that such an approach involves potential for application in the classroom to provide a relevant role for visualization and learning.

Keywords: number sequences, board and tiles, Padovan or Cordonier, mathematics teacher, visualization

INTRODUCTION

In the study of numerical sequences, authors of the history of mathematics books (Eves, 1969; Gullberg, 1997; Kleiner, 2012; Stakov, 2009) usually devote much attention to the case of the Fibonacci sequence, which is predominantly linked to the problem that made a particular set of numbers popular, when interpreted through the reproduction model of pairs of immortal rabbits. It is clear in the corresponding illustration, and it relates to the following numerical list: 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ... In **Figure 1**, for example, shows an anecdotal illustration Gulberg (1997) provided.

On the other hand, **Table 1** reveals examples that are usually disregarded in literature, such as the recurrences that characterize the extension of the Fibonacci sequence (tribonacci, tetranacci, pentanacci, etc.) as well as the recurrences that characterize the extension of the Padovan or Cordonier sequence. Nevertheless, considering the formal and naturally algebraic character of the notions and recurrences we indicate below, how could we explore, from a heuristic point of view, the visualization and intuition of mathematical properties related to the notion of numerical sequences?

Based on the previous question, specific properties related to the Padovan or Cordonier sequence, which have been chosen as the main objective of the present work, will be visualized, because it constitutes a sequence relatively overlooked by authors of mathematics history books. Furthermore, to explore visualization and perception of arithmetic and combinatorial properties, we will employ the notion of board and tiles introduced, in a pioneering way, by Benjamin and Quinn (2003), which provided an increase in research in several countries, however, research essentially developed in the field of pure mathematics.

In this sense, recalling that Benjamin and Quinn (2003) proposed the following laconic problem: Considering a board with dimensions 1 x n and tiles determined by a 1 x 1 square (light brown) and a 1 x 2 domino (light brown), in how many ways can one determine tilings using only a square and a domino? In **Figure 2** illustrates an n-board and the tiles considered by Benjamin and Quinn (2003). When considering the term fn that designates the number of tiles that take shape from a square and a domino, verify the following relationship:

$$f_n = F_{n+1}, n \geq 0 \quad (1)$$

From the previous relationship established by Benjamin and Quinn (2003), one infers that sets $\{f_n\}_{n\in IN}$ and $\{F_n\}_{n\in IN}$ coincide unless there is a reordering of the indices. **Figure 2** shows an n-board that corresponds to the Fibonacci sequence.

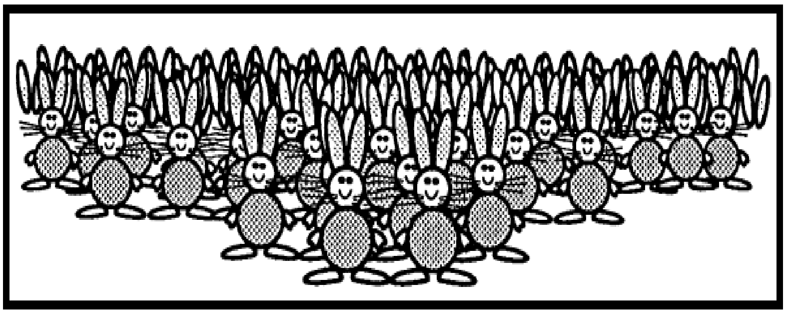

Figure 1. Anecdotal description of the reproduction of immortal rabbits (Gulberg, 1997)

Table 1. Recurrences related to the extension and generalization of Fibonacci and Padovan

Description of recurring number sequences	
Recurrence	Recurrence and initial values
Fibonacci	$F_n = F_{n-1} + F_{n-2}$, with $F_0 = 0, F_1 = 1$
Tribonacci	$F_n^3 = F_{n-1}^3 + F_{n-2}^3$, with $F_0 = 0, F_1 = 1$
Tetranacci	$F_n = F_{n-1} + F_{n-2}$, with $F_0 = 0, F_1 = 1$
....	....
Padovan or Cordonier	$P_n = P_{n-2} + P_{n-3}$, with $P_0 = 0, P_1 = 1, P_2 = 1$
Tridovan	$P_n = P_{n-2} + P_{n-3} + P_{n-4}$, with $P_0 = 0, P_1 = 1, P_2 = 1$
Tetravan	$P_n = P_{n-2} + P_{n-3} + P_{n-4} + P_{n-5}$, with $P_0 = 0, P_1 = 0, P_2 = 1, P_3 = 1$
Z -Dovan	$P_n = P_{n-2} + P_{n-3} + \cdots + P_{n-4} + P_{n-r}$, with$P_0 = 0, P_1 = 0, P_2 = 1, P_3 = 1$

Note. Prepared by the authors

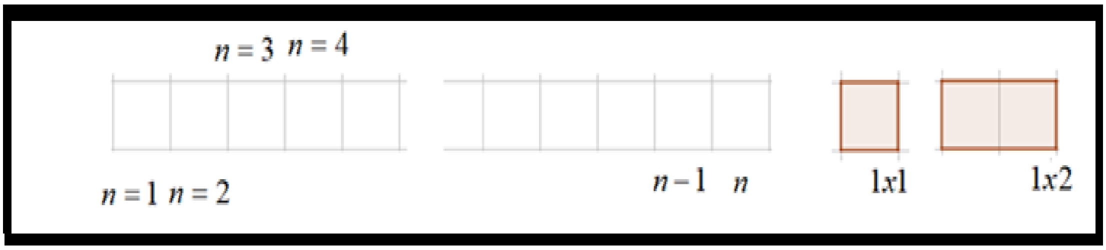

Figure 2. Description of the Fibonacci board and tiles (Benjamin & Quinn, 2003)

On the other hand, given the restricted nature of the discussion by authors of the history of mathematics books (Alves, 2017, 2024; Alves & Oliveira, 2024; Eves, 1969) and other essentially abstract and formal biases (Alves et al., 2024; Craveiro, 2004; Graham et al., 1994; Grimaldi, 2012), in the subsequent sections, we will address the case of the Padovan or Cordonier sequence (Alves & Catarino, 2022). We will show that the notion of board and combinatorial properties with tiles allows us to explore visualization from a heuristic and differentiated point of view, as we provide an increase in a necessary and strategic mathematical culture when we target the mathematics teacher.

LITERATURE REVIEW

Research on the notion of recurring numerical sequences gained greater repercussion after the work of Benjamin and Quinn (2003). With the origin and specific interest in the context of Pure Mathematics, we can see the advancement and interest in properties of generalized numerical sequences by numerous researchers and from Vieira et al. (2022). When we consult studies in Pure Mathematics and current research, we can see several cases of numerical sequences that do not yet have corresponding properties and representation via board and tiles. For example, we can consider the case of the sequence that determines properties related to Lichtenberg numbers and that are determined by the following recurrence:

$$Li_{n+1} = Li_n + 2Li_{n-1} + 1 \tag{2}$$

with the initial values defined by $Li_0 = 0, Li_1 = 1$.

More recently, Cerda-Morales (2025) describes other inductive formulas and abstract properties.

Soykan (2023a, 2023b, 2023c, 2023d) introduced several examples of new recurring number sequences and with interest in their generalized properties and a modern approach to the subject. From his work we came to know new properties, for example, of the Pandita, Richard, Pierre and Olivier numbers. All these categories of numbers are determined from numerical recurrences of second, third and fourth order.

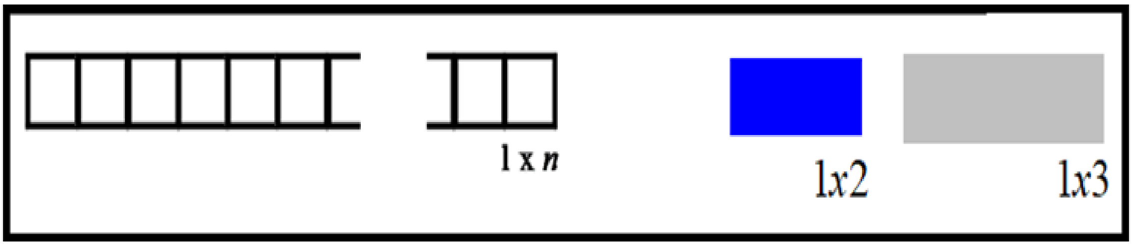

Figure 3. Description of the Padovan or Cordonier board according to Tedford (2019) (Source: Authors' own elaboration)

We can mention the example of Leonardo's sequence, whose modern and updated approach to the subject can be found in the work of Catarino and Borges (2019). In fact, in this work, we come across new mathematical properties resulting from the following recurrence:

$$Le_{n+1} = Le_n + Le_{n-1} + 1 \quad (3)$$

with the initial values defined by $Li_0 = 0, Li_1 = 1$.

After the work of Catarino and Borges (2019) we can verify the development of numerous other works involving the generalization of Leonardo's sequence, however, we did not identify its combinatorial interpretation via board and tiles.

We also recall the case of the repunity sequence, which is related to the Mersenne sequence. Indeed, considering the following recurrence defined by:

$$R_{n+1}(b) = (1 + b)R_n(b) - bR_{n-1}(b) \quad (4)$$

with the condition$R_0(b) = 0, R_1(b) = 1, b \geq 2$.

We can argue, for example, that this form of recurrence and properties are described by Snyder (1982). Easily, when we consider the case of b = 10 we can determine the set of repunit numbers and when b = 2 we can determine properties related to the Mersenne numbers. However, when we consult other more contemporary works on the subject, such as the work of Costa et al. (2025), we can see that only the abstract and structuring aspects in Pure Mathematics are contemplated and have a correlation with the notion of numerical sequence.

Given a scenario that tends to emphasize formal aspects regarding the notion of recurring numerical sequences, in the subsequent section, we will address a case that allows us to extract important repercussions for the training of Mathematics teachers.

PADOVAN OR CORDONIER BOARD AND PROPERTIES WITH TILINGS

In scientific literature, we find a growing interest in the study of properties of numerical sequences through the notion of board (Benjamin & Walton, 2009; Benjamin et al., 2008; Bodeen et al., 2014; Chinn et al., 2007; Dresden & Hopkins, 2021; Dresden & Ziqian, 2022).

In the **Figure 3**, a board with dimensions $1 \ x \ n$ and a set of tiles determined by domino (blue) $1 \ x$ 2 and a trimino (gray)1 x 3. From establishing these tiles, Tedford (2019) describes an n–board which we will call the Padovan board, due to an association and fundamental properties related to the Padovan or Cordonier sequence. From the n–board and fixing of the set of tiles, we will determine the corresponding tilings!

Figure 4 exemplifies the particular and initial cases $n = 0,1,2,3,4,5,6,7,8,9,10,11$. Unlike the case of the Fibonacci sequence, we can see that the numerical growth of the Padovan or Cordonier sequence is much slower. Furthermore, from the set of tiles, we can determine the following tiling count:

$$p_1 = 0, p_2 = 1, p_3 = 1, p_4 = 1, p_5 = 2, p_6 = 3, p_7 = 3, p_8 = 4, p_9 = 5, p_{10} = 7, p_{11} = 9, etc. \quad (5)$$

In **Figure 4**, we can visualize and carry out a direct count based on the number of tiles to which we match, for example, the behavior of the sums indicated by:

$$p_5 = 2 = 1 + 1 = p_3 + p_2, p_6 = 3 = 2 + 1 = p_4 + p_3, p_{11} = 9 = 5 + 4 = p_9 + p_4, etc. \quad (6)$$

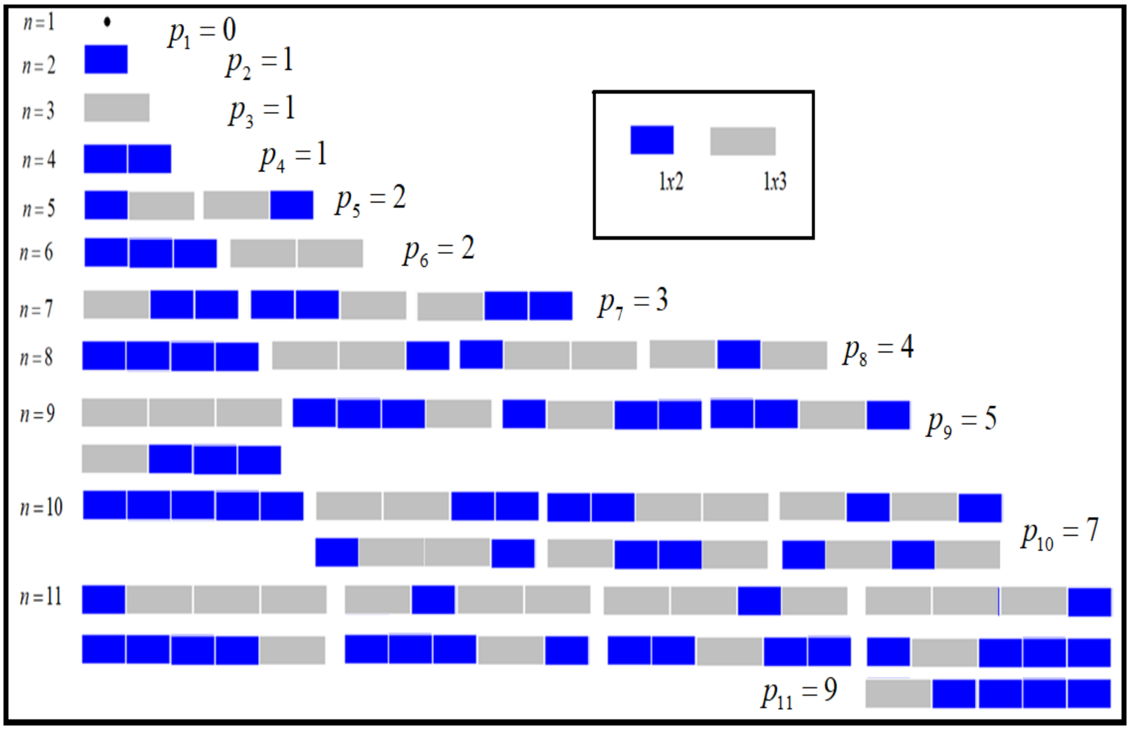

Figure 4. Particular examples of the tilings of the Padovan or Cordonier board (Source: Authors' own elaboration)

Next, we will present a theorem that formalizes specific ideas we indicated previously. Based on Tedford's work (2019), we can see two classes of tilings. A class of tilings that enclose in the order cells or a fixed position $(n-1,n)$, i.e., they end with a domino. A second class of tilings that enclose in the order cells or a position $(n-2,n-1,n)$, i.e., they end with a trimino. In Theorem 1, we indicate a recent result formulated by Tedford (2019), which provides the combinatorial interpretation of the Padovan or Cordonier sequence, in addition to allowing the verification of numerous identities through counting sets of tilings, such as:

$$p_n = p_{n-1} + p_{n-5}, p_n = p_{n-1} + p_{n-2} - p_{n-4}, p_n = p_{n-1} + p_{n-3} - p_{n-6}, n \geq 7 \tag{7}$$

Theorem 1

Considering an n-board with dimension $1xn$ and having fixed the set of tiles: (blue domino); (gray trimino). Being set $\mathfrak{I}_n$ of all tilings, of length n, when we use domino- (red) and trimino- (blue) type tiles, we have:

$$p_n = |\mathfrak{I}_n| = P_{n-2} \tag{8}$$

where $n \geq 1$ (Tedford, 2019).

Demonstration

Considering Tedford's arguments (2019), we will consider the set of elements $\mathfrak{I}_n$ that we determine from the tiles (dominoes, triminoes). Furthermore, considering some initial values present in the recurrence of set $\{P_n\}_{n\in IN}$, we establish that:

$$p_1 = |\mathfrak{I}_1| = 0 = P_{-1}, p_2 = |\mathfrak{I}_2| = 1 = P_0, p_3 = |\mathfrak{I}_3| = 1 = P_1 \tag{9}$$

Occur, and finally, another value starts with:

$$p_4 = |\mathfrak{I}_4| = 1 = P_2 \tag{10}$$

In these terms, when we examine the set of numbers $\{p_n\}_{n\in IN}$, we see that they admit the same values and initial conditions when compared to the set of Padovan numbers, indicated by $\{p_{n-2}\}_{n\in IN}$. Then, for the corresponding indexes for $n \geq 3$, we will fix two sets. The first set of tilings of length n that ends with a domino and which we will designate as D_n. The second set of tilings of length n and that ends with a trimino, which we will designate as T_n. Naturally, we will have:

$$D_n \cap T_n = \emptyset \tag{11}$$

Finally, when we consider the set of all tile contributions, i.e., when we take $D_n \cup T_n$, we can see that the total number of tiles p_n corresponds precisely to the sum indicated by:

Figure 5. Description and construction of the Padovan or Cordonier board with manipulative concrete materials (Vieira, 2024)

$$F_{n+1} - 1\ p_n = |D_n| \cup |T_n| = p_{n-2} + p_{n-3} \quad (12)$$

i.e., coinciding with the recurrence indicated in **Table 1**. Thus, we conclude that $p_n = P_n, n \geq 0$.

It should be noted that, in the set of Tedford's (2019) arguments, we can only consider one domino and one trimino. Naturally, when we consider the case of the Fibonacci sequence according to the problem proposed by Benjamin and Quinn (2003), we conclude that a square and a domino solve the problem and produce the desired relationship. On the other hand, we recall that Grimaldi (2012) determines another board with dimensions $2\ x\ n$ whose cells are made up of cells with similar dimensions $2\ x\ 1$ with the same orientation listed. Grimaldi (2012) fixes a set of vertical and horizontal tiles (dominoes). In turn, Craveiro (2004) uses another board formulation, with dimensions $2\ x\ n$, considering as tiles a white square $1x1$ and a blue square $2x2$. Likewise, we can identify relationships with Fibonacci. With similar reasoning, in the subsequent section, we will see another set of tiles that allow us to establish relations with the Padovan or Cordonier sequence.

PADOVAN OR CORDONIER BOARD AND PROPERTIES WITH TILINGS WITH COLOR PREFERENCE

In **Figure 5**, we present the Padovan or Cordonier board involving the use of linear tiles in the case of the Padovan sequence, and circular tiles in the case of the Perrin sequence. The board was made of concrete material, and on the left side, we can identify the colored tiles and, in particular, a black square ■ which will play an additional role in the rules of tiling composition, and in specific cases, we identify color preference and hierarchy. Although Vieira (2024) develops a new notion of circular board related to Perrin's recurrent sequence. In this work we will consider only linear boards.

When considering a set of tiles that we indicate in **Figure 6**, we formulate the addition of the following set of rules:

R1: For each integer n, the tiles that can fill each tiling are blue dominoes and gray triminos;

R2: For each tiling, the black square ■ $(1x1)$ can only occur at the beginning (i.e., in order cell 1) and will only occur once;

R3: For each integer $n \geq 0$, the product operation between tiles will be well defined only under the condition that the black square operation ■ $(1x1)$ results in a tiling of length less than or equal to n.

In **Figure 6**, we see an n–board and a set of fixed tiles: a black square, a blue domino, and a gray trimino.

Then, considering the set of rules $\{\Re_1, \Re_2, \Re_3\}$, we can determine the corresponding tilings. Indeed, **Figure 7** exemplifies the particular and initial cases $n = 0,1,2,3,4,5,6,7,8,9,10,11$. Unlike the case of the Fibonacci sequence, we can see that the numerical growth of the Padovan or Cordonier sequence is much slower. Furthermore, defining the term $p_n{}^*$, which corresponds to the number of tilings, with the set of tiles fixed, we can determine the number of tilings:

$$p_1{}^* = 1, p_2{}^* = 1, p_3{}^* = 2, p_4{}^* = 2, p_5{}^* = 3, p_6{}^* = 4, p_7{}^* = 5, p_8{}^* = 7, \textit{etc.} \quad (13)$$

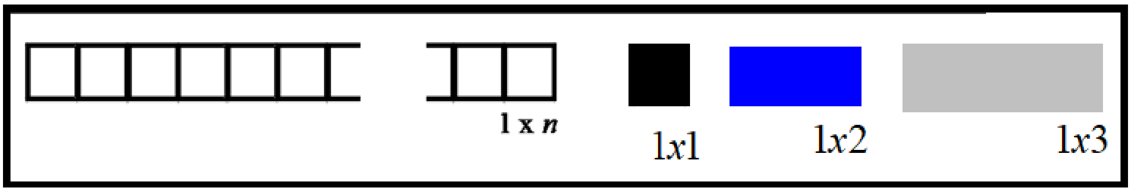

Figure 6. Description of the Padovan or Cordonier board according to Vieira (2024) (Source: Authors' own elaboration)

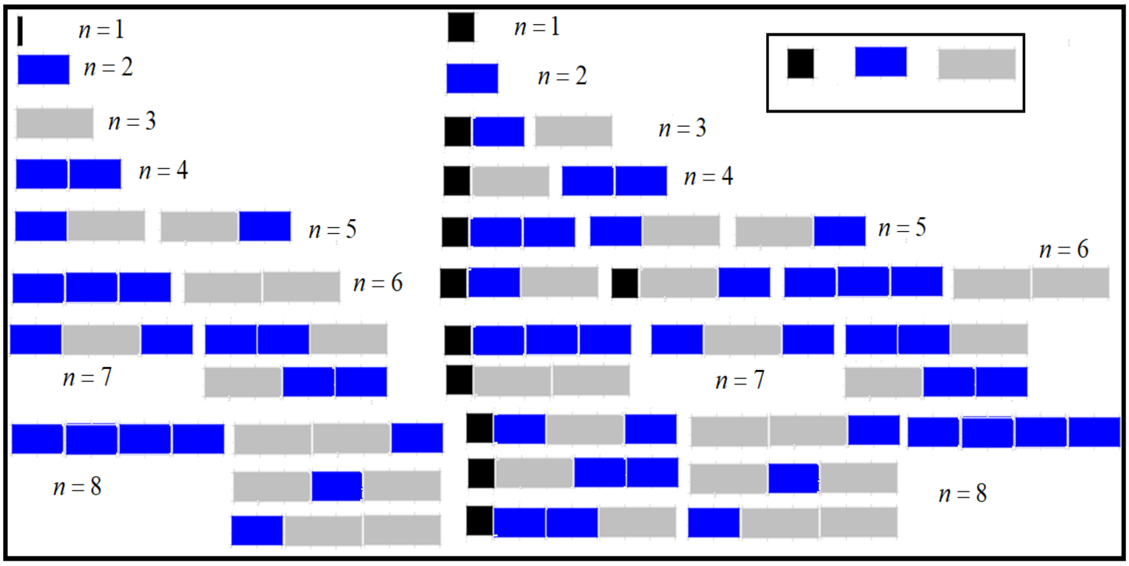

Figure 7. Particular examples of the tilings of the Padovan or Cordonier board (Source: Authors' own elaboration)

Then, considering the set of rules $\{\mathfrak{R}_1, \mathfrak{R}_2, \mathfrak{R}_3\}$, we can determine the relationships with the elements of the set $\{p_n^*\}_{n \in IN}$ and conclude that unless there is a reordering of indices, we will find a numerical correspondence with the Padovan or Cordonier sequence, which represents a modification of the theorem discussed by Vieira (2024).

Theorem 2

Considering an n-board with dimension $1xn$ and having fixed the set of tiles: ■ (black square); ■ (blue domino); ■ (gray trimino). Taking also the set of rules $\{\mathfrak{R}_1, \mathfrak{R}_2, \mathfrak{R}_3\}$ and defining the element p_n^* corresponding to the number of tilings of length n, when we use the above-mentioned tiles, then, we have:

$$p_n^* = P_n \; etc \tag{14}$$

with $n \geq 1$.

Demonstration

Given an integer $n > 0$ and an n-board. Setting the rules $\{\mathfrak{R}_1, \mathfrak{R}_2, \mathfrak{R}_3\}$ corresponding to the set of tiles indicated in the statement. We will proceed by induction on 'n'. Indeed, considering rule $\mathfrak{R}_1$, we can determine tilings from the domino and the trimino. Considering the term p_n^* designating the number of tilings defined by:

$$p_n^* = p_{n-1} + p_n, n \geq 1 \tag{15}$$

Let us note that in case $n = 0$, we will assume that:

$$p_0^* = 0 = P_0 \tag{16}$$

In case $n = 1$, we will have:

$$p_1^* = 0 + 1 = p_0 + p_1 = 1 = P_1 \tag{17}$$

In case $n = 2$, we will have:

$$p_2^* = 1 + 0 = 0 + 1 = p_1 + p_2 = P_2 \tag{18}$$

In case $n = 3$ we will have:

$$p_3^* = 1 + 1 = 2 = p_2 + p_3 = P_2 \tag{19}$$

Fixing indexes $(n-1, n, n+1)$, we observe that to determine the total number of tilings that correspond to the n–board, we consider the following sum:

$$p_n^* = p_{n-1} + p_n \tag{20}$$

and that the term p_{n-1} corresponds to the number of tilings corresponding to (n – 1) – board; however, the operation is carried out with the black square. In turn, term p_n corresponds to the number of tilings determined only by dominoes and triminos. Regarding our inductive hypothesis, we can write:

$$p_n^* = p_{n-1} + p_n = p_{n+2} \tag{21}$$

Considering that the elements of the set $\{p_n\}_{n \in IN}$ enjoy the property corresponding to the Padovan sequence. Furthermore, in view of Theorem 1, we will also have:

$$p_n^* = p_{n-1} + p_n = p_{n+2} = P_n, n \geq 0 \tag{22}$$

In the inductive step, we write:

$$p_{n+1}^* = p_n + p_{n+1} = p_{n+3} = P_{n+1} \tag{23}$$

i.e., we establish equality:

$$p_{n+1}^* = P_{n+1} \tag{24}$$

for every integer $n \geq 0$.

To conclude the current section, we point out an essentially formal and abstract component, in relation to which we find information on the recent advances in research around generalized numerical sequences (Alves & Catarino, 2022; Alves et al., 2024; Došlić & Podrug, 2022; Spreafico, 2014). On the other hand, as we emphasize in our work, it is essential to provide an alternative approach to access and increase the teacher's mathematical culture, especially through the notion of board and tiles, emphasizing visualization and understanding their properties (Hopkins, 2025).

CONCLUSIONS

In the preceding sections, we sought to highlight the role of visualization in the context of understanding specific arithmetic and combinatorial properties related, in particular, to the Padovan or Cordonier sequence. The relevance of addressing such a sequence is that it represents several properties that relate to the Fibonacci sequence in an almost hegemonic way and is often neglected by authors of the history of mathematics books (Alves, 2017) as we expand and consider a bigger set of examples of numerical sequences.

Using our guiding question as a reference, we indicate a heuristic point of view, emphasizing the visualization and intuition of mathematical properties related to number sequences. In these terms, through the notion of an n–board with tiles, by fixing a specific set of tiles in the context of pure mathematics and a learning scenario and development of practical activities with mathematics teachers (see **Figure 8**), we can explore numerous arithmetic and combinatorial properties involving identities and the corresponding verification of theorems, as we addressed in the cases of Theorem 1 and Theorem 2.

Finally, the notion of representation of properties of numerical sequences allows a broad extension scenario and the search for examples of new numerical sequences recently introduced in scientific literature whose formulation we have not come across via a board with tiles until now. Thus, this perspective can potentially drive new research into the education of mathematics teachers, as it plays a relevant role in visualization.

Contribution to the Literature and Future Directions

Based on some limitations, several recommendations and future directions are suggested for further research. In fact, we can see a wide list of recurring numerical sequences that do not yet have a corresponding representation via Board and tiles and that these properties allow and enable the development of a wide range of activities involving the production and elaboration of concrete and manipulable materials and that involve properties of these numerical sequences.

Finally, in our future work, we will propose and work on learning scenarios with Mathematics teachers that involve the generalization of the Board proposed by Benjamin and Quinn (2003). In **Figure 9**, we can have varied dimensions and, in a generalized way, several cases of tiles and different colors. The learning situation consists of stimulating the diversity of properties, dimensions and colors and that teachers can make correspondences or not with the Fibonacci sequence, for example. In **Figure 9** we indicate some examples aiming to clarify their application and use in the classroom.

Figure 8. Activities developed by Vieira (2024), involving the Fibonacci board (on the left side) and the Padovan or Cordonier board (on the right side) (Vieira, 2024)

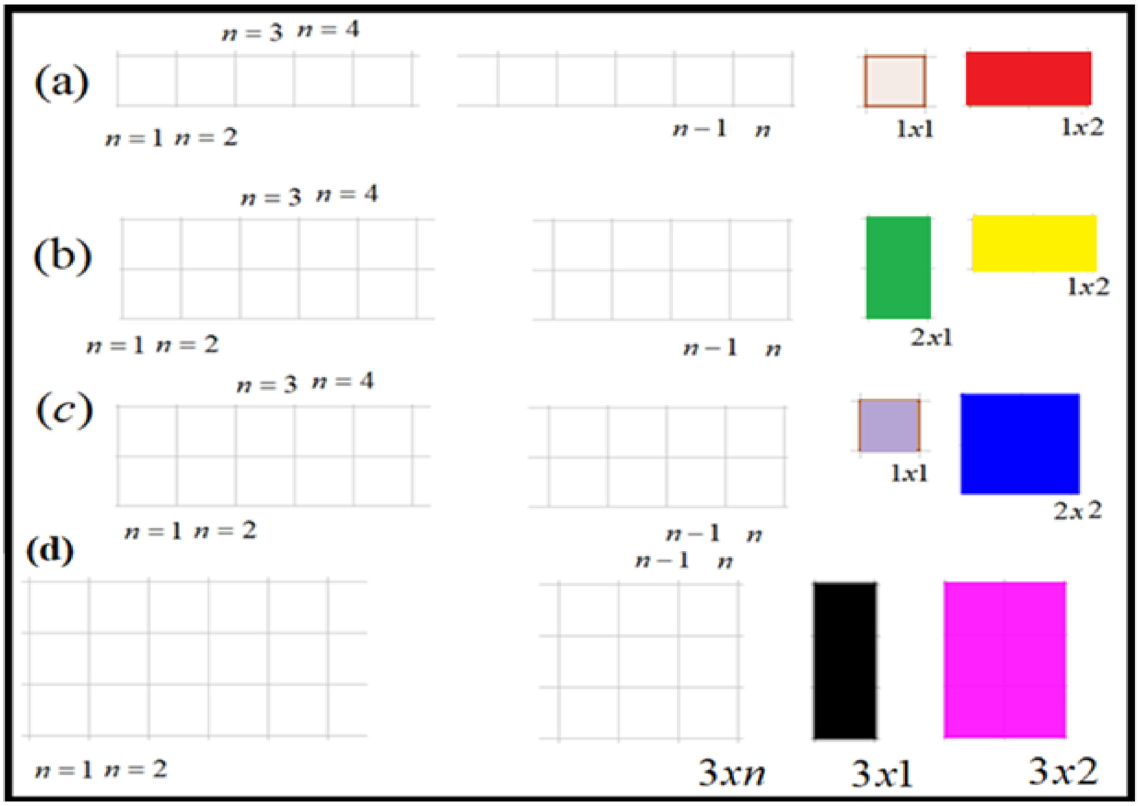

Figure 9. Description of various boards and tiles, with different dimensions and colors related to the notion of recurring numerical sequence (Source: Authors' own elaboration)

Author contributions: FA, RV, PC, & MM: data curation, visualization, testing and investigation, writing – original draft, writing - review & editing. All authors have agreed with the results and conclusions.

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AI statement: The authors stated that the scientific work does not use Generative AI or AI-based tools.

Declaration of interest: No conflict of interest is declared by the authors. No AI technologies were used in any part of this study.

Data sharing statement: Data supporting the findings and conclusions are available upon request from the corresponding author.

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